AER210: Vector Calc
Midterm Review

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Disclaimer: I am skipping over a lot of the formalities. A lot of the theorems cited rely on certain conditions (i.e. continuity, differentiable). However, they should all work on "nice" looking functions, so I left them out.

1 Multiple Integrals

Multiple integrals are used when integrating over regions or volumes. Just like Clairut's Theorem, we can swap the order:

$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

This is not the case for general regions. In general (if we look at 3D case), we can write

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{2}$$

where the region E can be defined as

$$E = \{(x,y,z) | a \le x \le b,\, g_1(x) \le y \le g_2(x),\, u_1(x,y) \le z \le u_2(x,y)\}. \tag{3}$$

Other region types involve just permutations.

1.1 Coordinate Systems

Cylindrical Coordinates

We can convert from cylindrical to rectangular coordinates

$$x = r\cos\theta,\; y = r\sin\theta,\; z = z \tag{4}$$

And triple integral is given by

$$\iiint f(x,y,z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \tag{5}$$

Spherical Coordinates

We can convert from spherical to rectangular coordinates (should be polar when $\phi = \pi/2$)

$$x = \rho\cos\theta\sin\phi,\; y = \rho\sin\theta\sin\phi,\; z = \rho\cos\phi \tag{6}$$

and the integral is given by

$$\iiint f(x,y,z)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{7}$$

1.2 Change of Basis

In 1D calculus, we performed u-substitutions as follows. If $x = f(u)$. Then $\mathrm{d}x = f'(u)\,\mathrm{d}u$ and

$$\int_{u(a)}^{u(b)} f(x(u))\,(f'(u)\,\mathrm{d}u) \tag{8}$$

The same formula applies in multiple dimensions if we treat x and u as vectors, such that $f'(x)$ is the determinant of a *matrix of partial derivatives*, known as the **Jacobian**, where given $x = g(u, v)$ and $y = h(u, v)$, we have:

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \tag{9}$$

and thus

$$\iiint_R f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_E f(x, y, z) J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \tag{10}$$

where S is the same region as R but written in terms of u, v, w.

1.3 Surface Area

The surface area of a region D is given by

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}A \tag{11}$$

2 Div Grad Curl and All That

2.1 Line Integral

Suppose we define a line using the parametric equations $x(t), y(t)$ and there is a function f that acts on this line C, then:

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{x'^2 + y'^2}\,\mathrm{d}t \tag{12}$$

Alternatively, we can write

$$\int_C f(x, y)\,\mathrm{d}x = \int_a^b f(x(t), y(t)) x'(t)\,\mathrm{d}t \tag{13}$$

The line integral of a vector field F that acts on a curve C is given by

$$\int_C F \cdot \mathrm{d}r = \int_a^b F(r(t)) \cdot r'(t)\,\mathrm{d}t = \int_C F \cdot T\,\mathrm{d}s \tag{14}$$

where T is the unit tangent vector.

Fundamental

The fundamental theorem says that

$$\int_C \nabla f \cdot \mathrm{d}r = f(r(b) - r(a)) \tag{15}$$

Therefore, if we are given a line integral and the vector field can be written as the gradient of a function, then it is **conservative** and we can apply this theorem.

When is a vector field conservative? If $F(x, y) = (P(x, y), Q(x, y))$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{16}$$

and the converse holds for open simply-connected regions (i.e. no weird stuff happening). It is also conservative if $\nabla \times F = 0$.

2.2 Green's Theorem

We have another shortcut to calculate closed line integrals:

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A \tag{17}$$

Note that this is zero when the field is conservative. We can write Green's Theorem in vector form. Given $F = (P, Q)$ as before, we can write

$$\oint_C F \cdot \mathrm{d}r = \iint_D (\nabla \times F) \cdot \hat{k}\,\mathrm{d}A \tag{18}$$

2.3 Divergence and Curl

Define the operator $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^T$ such that

$$\text{div}F = \nabla \cdot F \tag{19}$$

and

$$\text{curl}F = \nabla \times F \tag{20}$$

Note that div curl $\nabla \cdot (\nabla \times F) = 0$ (we had a similar expression in linear algebra) and curl grad$F = 0$ (a block placed on a mountainous hill will slide down without rotating)

3 Parametric Surfaces

We can represent a 1D using a single parameter t. Similarly, we can represent a 2D surface using two parameters u, v. The overall idea is that we can define a surface as

$$r(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{21}$$

3.1 Surfaces of Revolution

We can parametrize a surface which was a result of revolution via:

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \tag{22}$$

3.2 Tangent Planes

The unit vector r_v, which points in the direction we move in if we *only* vary the parameter v is given by

$$r_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \tag{23}$$

and similarly for r_u. The plane can then be represented by the normal vector

$$r_u \times r_v \tag{24}$$

which should be nonzero for smooth surfaces.

3.3 Surface Area

The surface area is given by

$$A(S) = \iint_D |r_u \times r_v|\,\mathrm{d}A \tag{25}$$

3.4 Surface Integrals

We can generalize the previous result to the general case (i.e. a function f acts on a region S):

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(r(u,v))|r_u \times r_v|\,\mathrm{d}A \tag{26}$$

Note the similarity between this form and the similar form when we consider a function $z = g(x,y)$. We have

$$r_x \times r_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) \tag{27}$$

and

$$|r_x \times r_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \tag{28}$$

so we can easily convert between the two forms.

3.5 Oriented Surfaces

Similar to the above discussion, we can write the oriented normal surface in two ways:

$$n = \frac{r_u \times r_v}{|r_u \times r_v|} = \frac{-\frac{\partial g}{\partial x}\hat{\imath} - \frac{\partial g}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} \tag{29}$$

3.6 Surface Integrals of Vector Fields

The flux of $\boldsymbol{F}$ across S is given by

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\,\mathrm{d}S = \iint_D \boldsymbol{F} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\,\mathrm{d}A \tag{30}$$

4 Adolescent Level Calculus

4.1 Stoke's Theorem

If F is a vector field, then

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S} \tag{31}$$

This is just a three-dimensional version of Green's Theorem.

4.2 Divergence Theorem

Stoke's Theorem tells us that the curl of a function $\boldsymbol{F}$ on a surface $\boldsymbol{S}$ can be represented by how $\boldsymbol{F}$ interacts with the boundary of the surface.

Divergence Theorem tells us the same thing, but going from 3D to 2D:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_E \nabla \cdot \boldsymbol{F}\,\mathrm{d}V \tag{32}$$