CIV102: Problem Set #8

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November 18, 2020

1 Problem One

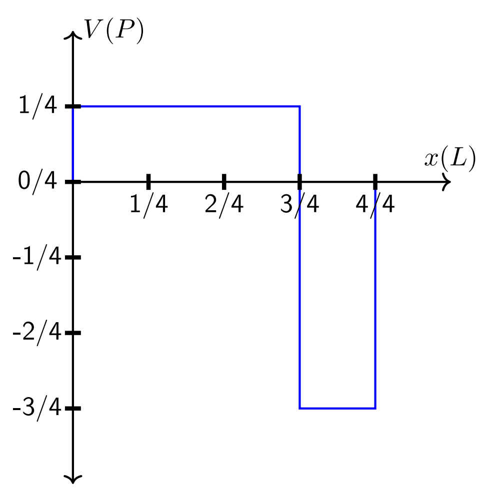
(a) We first perform the analysis for the beam on the left. Let the two reaction forces be A and C. Note that there are no horizontal forces from A since the net horizontal force is zero. Then:

$$A + C = P \tag{1}$$

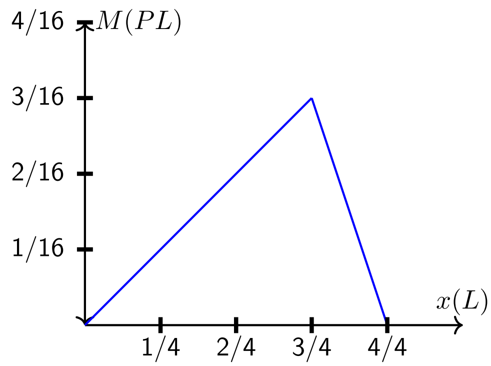
and balancing torques:

$$P\left(\frac{3L}{4}\right) = C\,(L) \implies C = \frac{3}{4}P \tag{2}$$

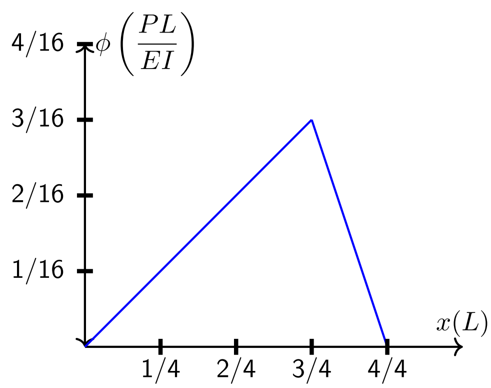
and similarly $A = \frac{1}{4}P$. The shear force diagram can be drawn as:



and the moment diagram:



and the curvature diagram:



The deflection at B with respect to the tangent line at A is:

$$\delta_{BA} = \bar{x}_{BA} \int_B^A \phi(x)\,\mathrm{d}x = \left(\frac{1}{3}\frac{3L}{4}\right)\left(\frac{1}{2}\frac{3L}{4}\frac{3}{16}\frac{PL}{EI}\right) = \frac{9}{512}\frac{PL^3}{EI} \tag{3}$$

The deflection at C with respect to the tangent line at A is:

$$\delta_{CA} = \left(\frac{2}{3}\frac{L}{4}\right)\left(\frac{1}{2}\frac{L}{4}\frac{3}{16}\frac{PL}{EI}\right) + \left(\frac{L}{4} + \frac{1}{3}\frac{3L}{4}\right)\left(\frac{1}{2}\frac{3L}{4}\frac{3}{16}\frac{PL}{EI}\right) = \left(\frac{1}{256} + \frac{9}{256}\right)\frac{PL^3}{EI} = \frac{5}{128}\frac{PL^3}{EI} \tag{4}$$

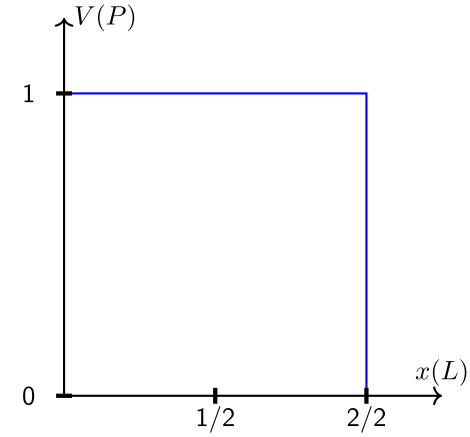
The angle the beam makes at A is given by:

$$\theta_A \cong \frac{5}{128}\frac{PL^2}{EI} \tag{5}$$

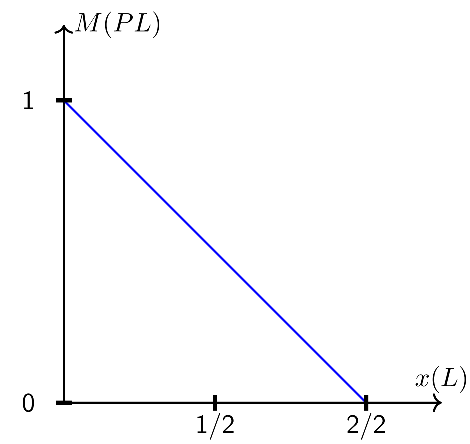
so the deflection of B with respect to the initial position is:

$$\Delta_B = \frac{3}{4}L\theta_A - \delta_{BA} = \frac{15}{512}\frac{PL^3}{EI} - \frac{9}{512}\frac{PL^3}{EI} = \frac{3}{256}\frac{PL^3}{EI} \tag{6}$$

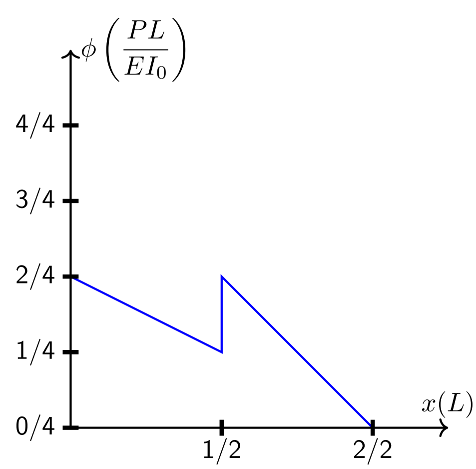
(b) Again, since there are no net horizontal forces, the reaction force at the left end is completely vertical, which we denote as A which is equal and opposite to P. It also exerts a counterclockwise moment with a magnitude equal to $M_A = PL$. Therefore, the shear force and moment diagrams are:



and the moment diagram:



For the curvature diagram, we have:



where the varying moment of inertia was taken into consideration. The deflection of point C is then given as:

$$\delta_{CA} = \bar{x}_1 A_{\text{right triangle}} + \bar{x}_2 A_{\text{rectangle}} + \bar{x} A_{\text{top left triangle}} \tag{7}$$

where we have broken up the curvature diagram into three shapes by breaking the trapezoidal shape into a rectangle and a triangle. Thus:

$$\delta_{CA} = \left(\frac{2}{3}\frac{L}{2}\right)\left(\frac{1}{2}\frac{L}{2}\frac{1}{2}\frac{PL}{EI_0}\right) + \left(\frac{L}{2} + \frac{1}{2}\frac{L}{2}\right)\left(\frac{L}{2}\frac{1}{4}\frac{PL}{EI_0}\right) + \left(\frac{L}{2} + \frac{2}{3}\frac{L}{2}\right)\left(\frac{1}{2}\frac{L}{2}\frac{1}{4}\frac{PL}{EI_0}\right) \tag{8}$$

$$= \left(\frac{1}{24} + \frac{3}{32} + \frac{5}{96}\right)\frac{PL^3}{EI_0} \tag{9}$$

$$= \frac{3}{16}\frac{PL^3}{EI_0} \tag{10}$$

Since A is a horizontal tangent, the displacement of point C is:

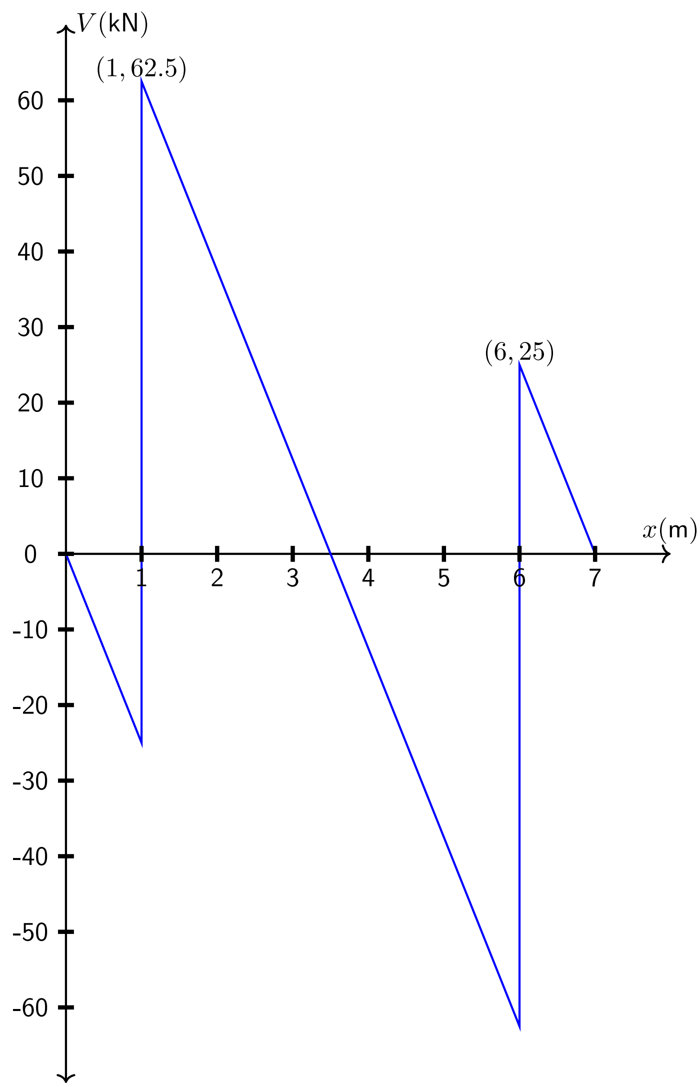
$$\Delta_C = \frac{3}{16}\frac{PL^3}{EI_0} \tag{11}$$

2 Problem Two

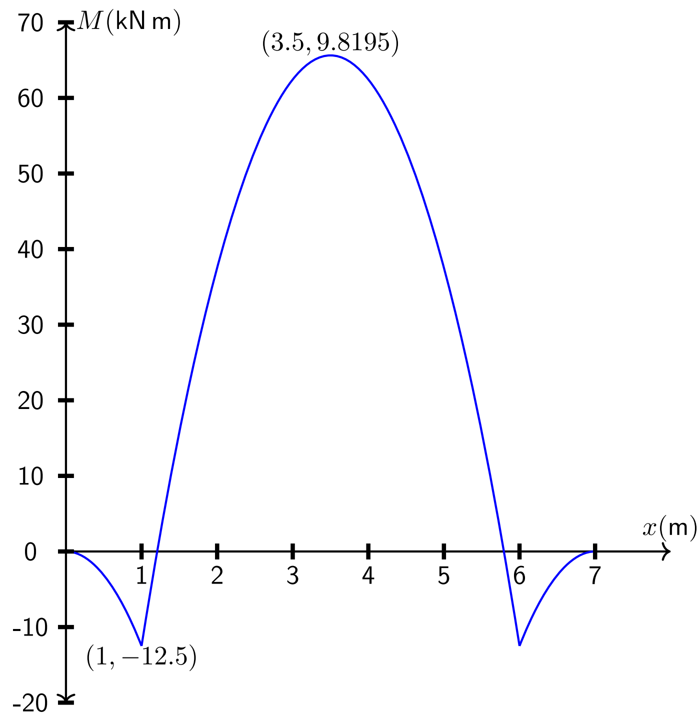
We can find the force F exerted by the supports due to symmetry. We have:

$$25(7) = 2F \implies F = 87.5\text{kN} \tag{12}$$

The shear force diagram looks like:



The moment diagram looks like:



Note that the intercepts are at $(1.209, 0)$ and $(5.791, 0)$, which represent locations where the moment is zero. The center of mass of the beam is:

$$\bar{y} = \frac{(150)(125 \cdot 300) + (350)(800 \cdot 100)}{800 \cdot 100 + 125 \cdot 300} = 286.17\text{mm} \tag{13}$$

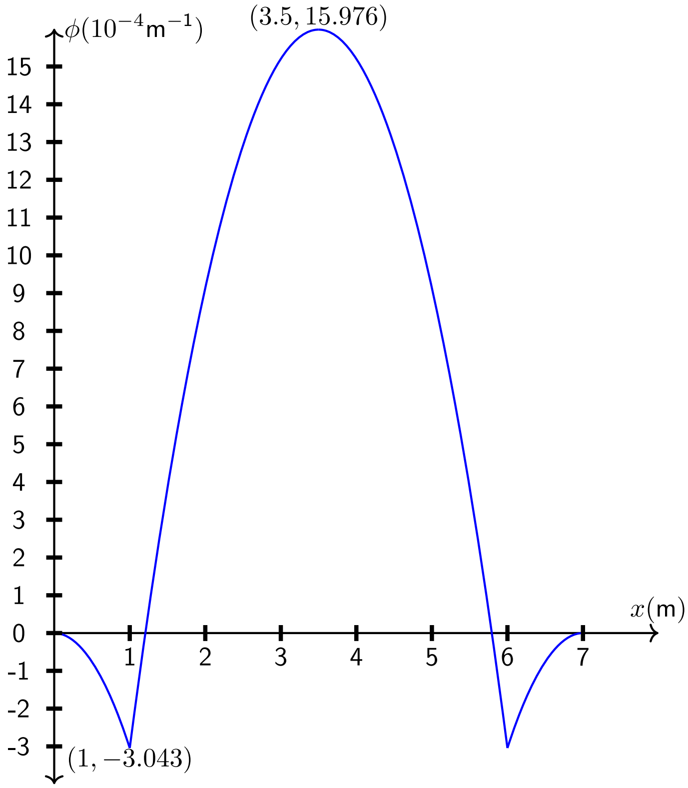
from the bottom. The moment of inertia is then:

$$I = \frac{1}{12}(800)(100)^3 + (800 \cdot 100)(350 - 286.17)^2 + \frac{1}{12}(125)(300)^3 + (125 \cdot 300)(286.17 - 150)^2 = 1369.19 \times 10^6\text{mm}^4 \tag{14}$$

Therefore, the flexural stiffness is:

$$EI = 4.1076 \times 10^4\text{kPa m}^2 \tag{15}$$

and the curvature diagram is simply scaled by the inverse of this:



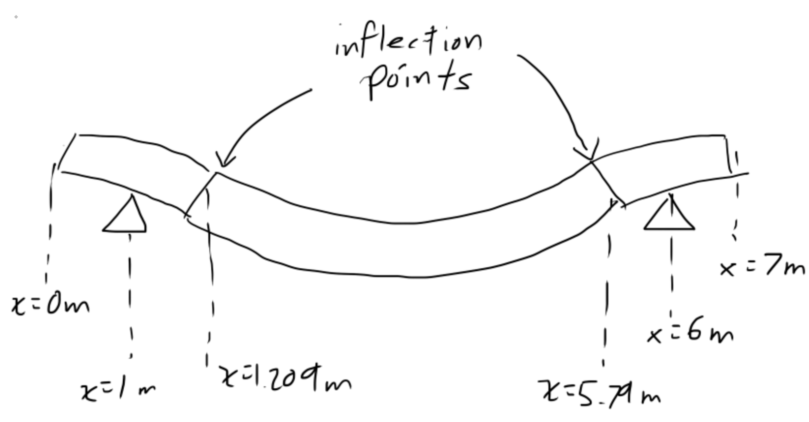
We can determine the equation of the curve in the middle to be:

$$\phi(x) = -3.043 \cdot 10^{-4} (x - 3.5)^2 + 15.976 \cdot 10^{-4} \tag{16}$$

We can calculate the deflection at point C to be:

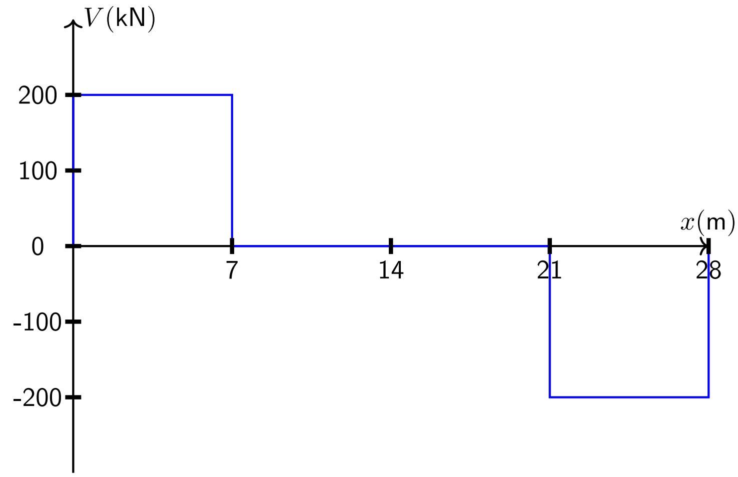
$$\delta_{C,mid} = \int_C^{\text{mid}} = \int_{3.5}^{6} (6 - x) \left(-3.043 \cdot 10^{-4} (x - 3.5)^2 + 15.976 \cdot 10^{-4} \right) \mathrm{d}x = 4.002\text{mm} \tag{17}$$

Since the slope at the midpoint is zero, we don't have to do anything else. The beam looks like this:

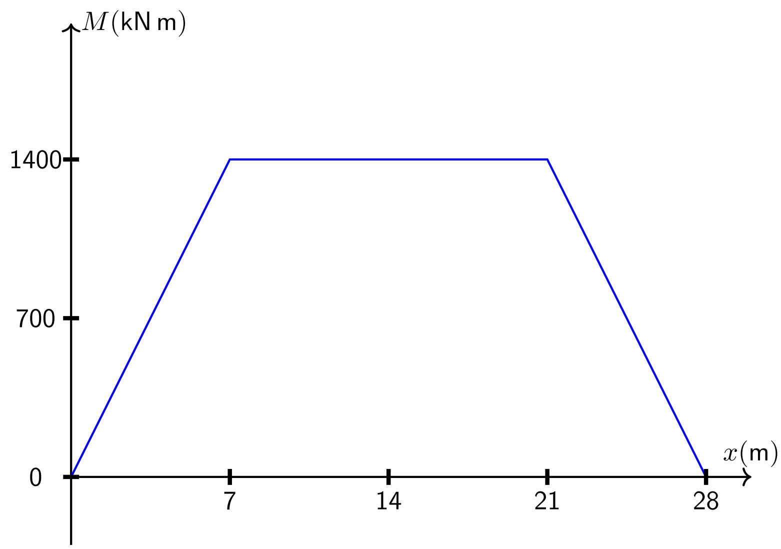


3 Problem Three

(a) If we ignore P_B, then the forces exerted by the two supports A and C as 200kN each, making the shear force diagram:



and the moment diagram:

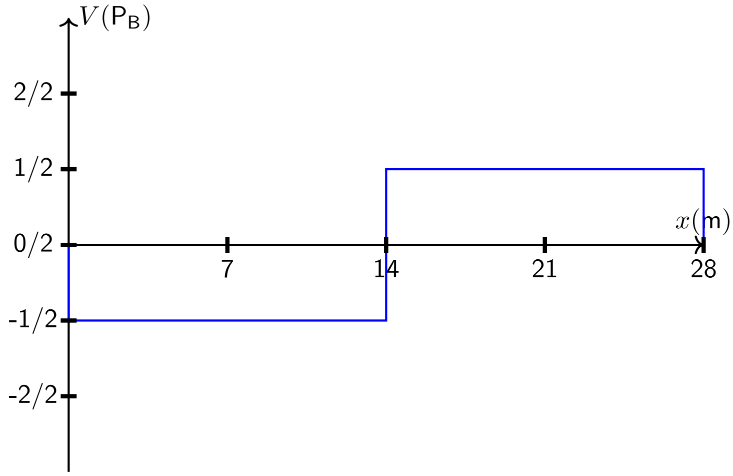


We can abuse symmetry here to determine the displacement of point C relative to B as:

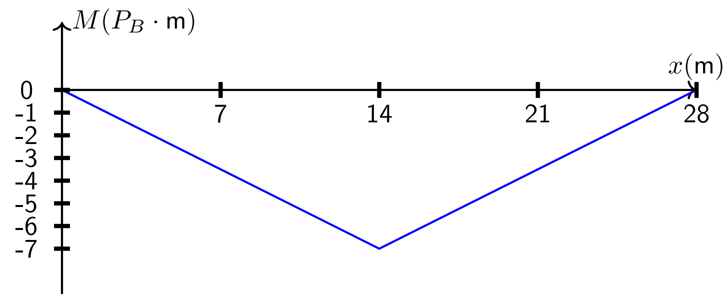
$$\delta_{CB} = \Delta_B = \frac{1}{EI}\left(\frac{2}{3}(7)\right)\left(\frac{1}{2}(7)(1400)\right) + \left(7 + \frac{7}{2}\right)(7 \cdot 1400) = 628.(83)\text{mm} \tag{18}$$

which is true since the horizontal tangent at the midpoint is zero.

(b) Due to symmetry, the two supports would each exert a force of $P_B/2$. Therefore, the shear force diagram looks like



and the bending moment diagram:



Abusing symmetry once again, the displacement of C relative to B is:

$$\delta_{CB} = 628.83 = \frac{1}{EI}\left(\frac{2}{3}(7000)\right)\left(\frac{1}{2}(14000^2)P_B\right) = 2.2866P_B \tag{19}$$

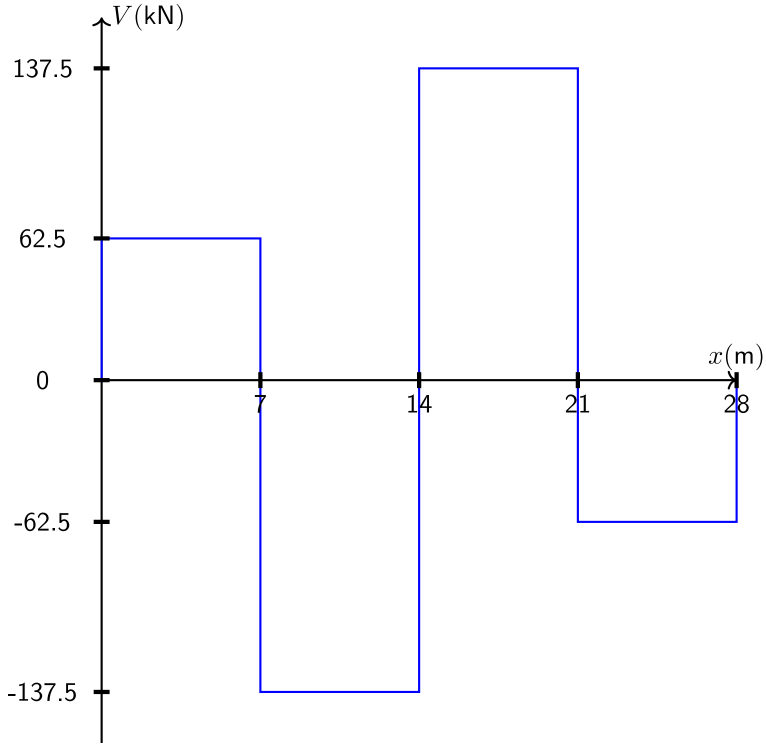
and solving for P_B gives:

$$P_B = 275\text{kN} \tag{20}$$

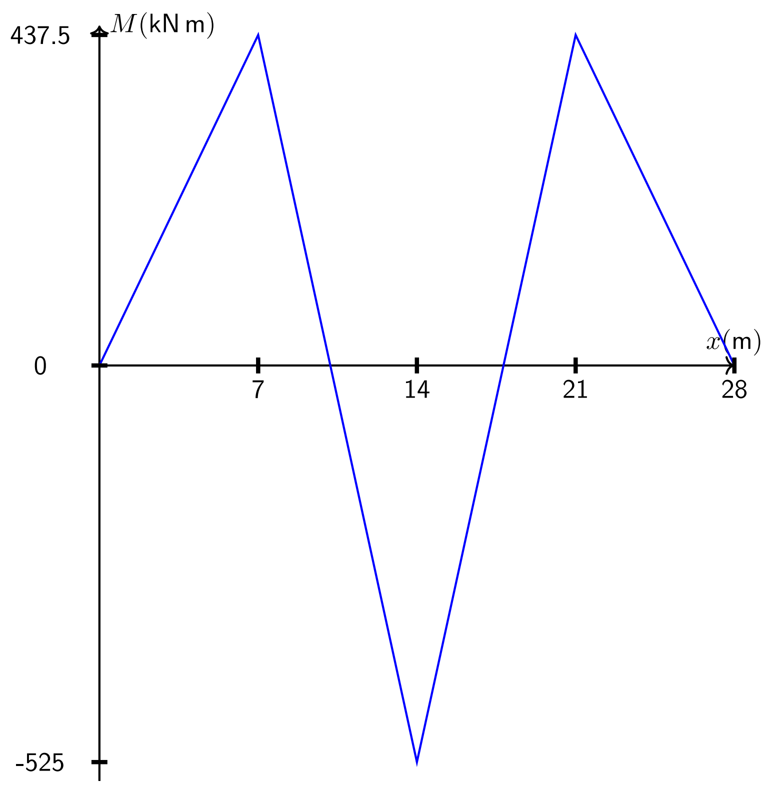
(c) With $P_B = 275$kN, the forces the two supports exert is:

$$A_y = C_y = \frac{1}{2}(200 + 200 - 275) = 62.5\text{kN} \tag{21}$$

such that the shear force diagram is:



and the shear force diagram as:



4 Problem Four

The maximum shear force occurs at the ends and it has a magnitude of:

$$V = \frac{1}{2}\left(5.5\text{kN m}^{-1} \cdot 3.5\text{m}\right) = 9.625\text{kN} \tag{22}$$

The maximum shear stress occurs at the center of the beam, which has a moment of area equal to:

$$Q = Ad = \left(90 \cdot \frac{180}{2}\right)\left(\frac{1}{2}180 - \frac{1}{2}\frac{180}{2}\right) = 364500\text{mm}^3 \tag{23}$$

The moment of inertia is equal to:

$$I = \frac{1}{12}(90)(180^3) = 43.74 \times 10^6\text{mm}^4 \tag{24}$$

such that the maximum shear stress is:

$$\tau = \frac{VQ}{Ib} = \frac{(9625)(364500)}{(43.74 \times 10^6)(90)} = \boxed{0.891\text{MPa}} \tag{25}$$

We can generalize equation 23 to:

$$Q = Ad = (90 \cdot y)\left(\frac{180}{2} - \frac{y}{2}\right) = 45y\,(180 - y) \tag{26}$$

and the shear stress distribution looks like this: