ESC194: Calculus Notes

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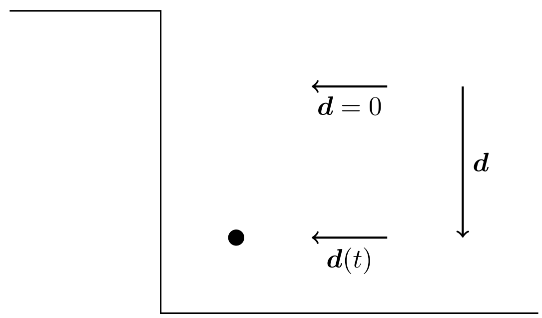
1 Introduction to Limits: The Motivation

Idea: The goal of the first few lectures is to define the derivative of a function *rigorously* **logically**. There are similar, but not quite the same, problems to the above:

- Defining the *slope* of a tangent to the curve
- Defining *speed* at an *instant*.

However, these concepts are not very well defined.

- Consider a falling object where a ball starts at a height of $d = 0$ and ends at a distance of $d(t)$.



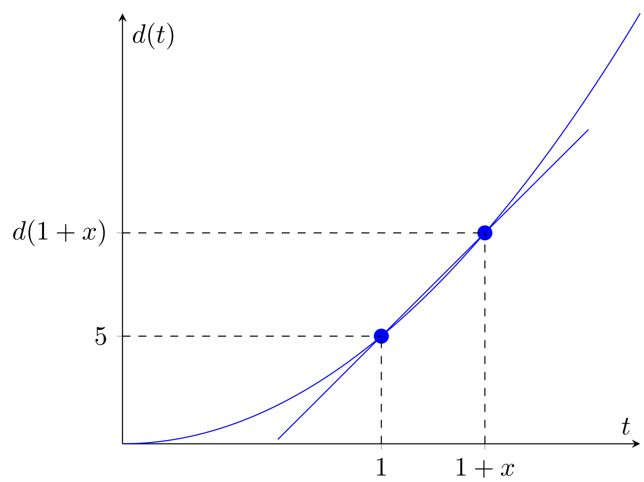
Suppose we are given the function $d(t) = 5t^2$ where d is in meters and t is in seconds. To determine the speed at the *instant* $t = 1$ sec, we can approximate using a secant-line:

$$m_{\text{sec}} = \frac{5(1.1)^2 - 5(1)^2}{1.1 - 1} = 10.5 \text{ m/s} \tag{1}$$

$$m_{\text{sec}} = \frac{5(1.01)^2 - 5(1)^2}{1.01 - 1} = 10.05 \text{ m/s} \tag{2}$$

The instantaneous speed *appears* to approach 10 m/s **exactly** at $t = 1$ sec, but we *cannot* be sure.

- We can do better by introducing a new variable $x[s]$ which represents the interval:



We then define a function $f(x)$ as

$$f(x) \equiv \frac{d(1+x) - d(1)}{x} \tag{3}$$

$$= \frac{5(1+x)^2 - 5(1)^2}{x} \tag{4}$$

$$= \frac{10x + 5x^2}{x} \tag{5}$$

$$= (10 + 5x)\left(\frac{x}{x}\right) \tag{6}$$

Let us *assume* (incorrectly) that $\dfrac{x}{x} = 1$ for all x. Then:

$$f(x) = 10x + 5 \tag{7}$$

So we could define the "speed at $t = 1$ s" as $f(0)$ which is exactly 10.

Warning: However: $\dfrac{x}{x} \neq 1$ when $x = 0$ as division by zero is not allowed as a legitimate operation of arithmetic. To explain why, we need to rigorously introduce numbers as defined by **axioms**.

- Example: For every $x \neq 0$ there is another number $\dfrac{1}{x}$ defined **implicitly** by $x \cdot \dfrac{1}{x} = 1$.

An implicit definition is when a quantity we are defining does not appear by itself on one side of the equation. Implicit definitions are used for the most basic axioms. If this was not true, then we can contradict ourselves. We would have:

$$0 \cdot \frac{1}{0} = 1 \tag{8}$$

but we can also prove that $0 \cdot a = 0$ for all numbers a, so if $\dfrac{1}{0}$ is a number, then $0 = 1$.

Furthermore, it is not possible to define $\dfrac{1}{0}$ if it was a number. A naive approach may be to define it as infinity, but infinity is not a number! If it was, then:

$$1 + \infty = \infty \tag{9}$$
$$2 + \infty = \infty \tag{10}$$
$$1 = 2 \tag{11}$$

Just because you **write** some symbol does not mean it **exists** as a number.

The correct expression for $f(x)$ is instead:

$$f(x) = \begin{cases} 10 + 5x, & x \neq 0 \\ \text{DNE}, & x = 0 \end{cases} \tag{12}$$

While DNE is not a number, $f(x)$ is still a legitimate function. We don't *fix* functions.

- The **limit** can be used to satisfy our intuitive feeling that the answer is exactly 10 m/s. The expression:

$$\lim_{x \to 0} f(x) \tag{13}$$

is a number.

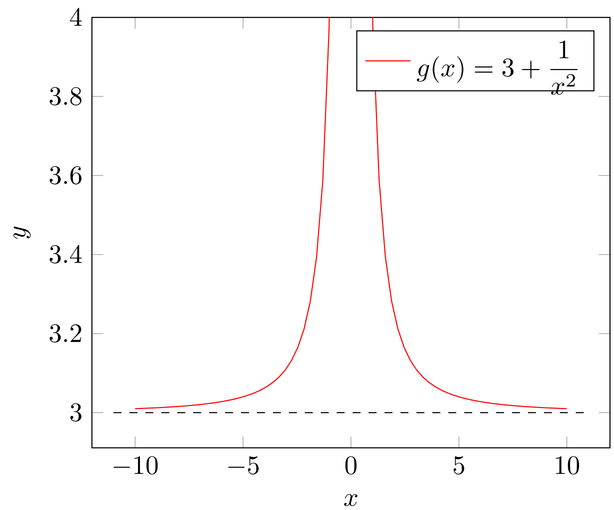
Warning: We can't always trust our intuition on whether a certain relationship will converge or diverge:

$$A = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{17} + \cdots \tag{14}$$
$$B = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{15}$$

The value of A will converge to 1, but the value of B will diverge[a] and will not exist. This is why we need a rigorous definition of what exactly "approaching" means.

[a] This is known as the harmonic sum.

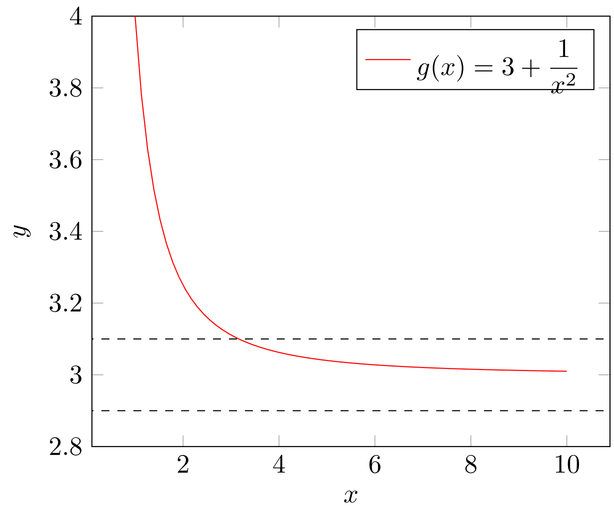
- A simpler type of limit considers what happens at infinity:



It appears that $g(x)$ approaches a value of 3 as x approaches infinity. The challenge is to find a **rigorous definition** of:

$$\lim_{x\to\infty} g(x) \tag{16}$$

so that we can prove it exists as a number. We want to be able to say that we can always find values of x large enough that the values of $g(x)$ will be as close as might be wanted to 3, say within 10^{-10} of 3. Geometrically, this is represented by always being able to pick an x great enough such that $g(x)$ is contained within the dashed lines.



We can solve this via trial and error. Say for a large $x_0 = 10^{100}$, we can show that for all

$$x > x_0 = 10^{100} \tag{17}$$

$$x^2 > 10^{200} \tag{18}$$

$$\frac{1}{x^2} < 10^{-200} \tag{19}$$

$$3 + \frac{1}{x^2} < 3 + 10^{-200} \tag{20}$$

Since $3 + 10^{-200} < 3 + 10^{-10}$, then the weaker case:

$$g(x) < 3 + 10^{-10} \tag{21}$$

must be true for all $x > x_0 = 10^{100}$ However, we need to find the lower bound as well. Next, note that:

$$g(x) = 3 + \frac{1}{x^2} > 3 > 3 - 10^{-10} \tag{22}$$

for all x. Therefore, for all $x > 10^{100}$, then: $3 - 10^{-10} < g(x) < 3 + 10^{-10}$.

- We can generalize this to any arbitrary bound. Some small number $\epsilon > 0$. We want to find some x_0 expressed in terms of ϵ such that for all $x > x_0$, the values of $g(x)$ will be within ϵ of 3.

> **Mnemonic**: The value of ϵ is imposed by the ϵnemy as a challenge.

Again, we use trial and error.

Example 1: Suppose we pick $x_0 = \frac{1}{\epsilon}$. Then for all $x > x_0 = \frac{1}{\epsilon}$, we have:

$$x^2 > \frac{1}{\epsilon^2} \tag{23}$$
$$\frac{1}{x^2} < \epsilon^2 \tag{24}$$
$$3 + \frac{1}{x^2} < 3 + \epsilon^2 \tag{25}$$
$$g(x) < 3 + \epsilon^2 \tag{26}$$
$$\tag{27}$$

which doesn't quite work since in order for the value to be within the bounds, the following must be true:

$$g(x) < 3 + \epsilon^2 \leq 3 + \epsilon \tag{28}$$

which is only true for $\epsilon \leq 1$ and is not true for all values of ϵ.

Example 2: Suppose we pick $x_0 = \frac{1}{\sqrt{\epsilon}}$. Then for all $x > x_0 = \frac{1}{\sqrt{\epsilon}}$, we have:

$$x^2 > \frac{1}{\epsilon} \tag{29}$$
$$\frac{1}{x^2} < \epsilon \tag{30}$$
$$3 + \frac{1}{x^2} < 3 + \epsilon \tag{31}$$
$$g(x) < 3 + \epsilon \tag{32}$$
$$\tag{33}$$

which provides the correct upper bound!

As a result, the value of $y = 3$ passes this challenge test, so we can define a new number:

$$\lim_{x\to\infty} g(x) \tag{34}$$

and assign it to the value of 3.

> **Idea**: There are three steps to find the limit:
> 1. Assume that $\lim_{x\to\infty}$ exists and guess a value for it.
> 2. Show that your guess passes a "challenge" imposed by ϵ
> 3. If you succeed then we can take that $\lim_{x\to\infty} g(x)$ exists as a number since we can assign it to your original guess.

- Similarly, for our original function $f(x)$ we can use a similar way to define the limit as $x \to 0$.but

2 The Fundamental Axioms

- Numbers are *bases* elements of mathematics, so they cannot be defined explicitly in terms of anything more basic.
- Instead they are defined *implicitly*, by imposing the rules, or **axioms**, that we require they satisfy.

> **Idea**: The axioms are *inspired* by physical reality, but are not *dictated by it*. They do not exist

- It is important to have as few axioms as possible (to make it philosophically more "pure", and to reduce the risk of contradictions:

  1. **Commutative Law**: For each pair $x, y \in \mathrm{Re}$,
  $$x + y = y + x \tag{35}$$
  and
  $$xy = yx \tag{36}$$
  2. **Associative Law**: For each triple $x, y, z \in \mathrm{Re}$,
  $$x + (y + z) = (x + y) + z \tag{37}$$
  and
  $$(xy)z = z(yz) \tag{38}$$
  3. **Distributive Law** For each triple $x, y, z \in \mathrm{Re}$,
  $$x(y + z) = zy + yz \tag{39}$$
  and
  $$(x + y)z = xz + yz \tag{40}$$
  4. **Existence of Identities**: There exists two distinct real numbers, denoted by 0 and 1 for which:
  $$x + 0 = 0 + x = x \tag{41}$$
  and
  $$x \cdot 1 = 1 \cdot x = x \tag{42}$$
  for each $x \in \mathrm{Re}$.
  5. **Existence of inverses** For each $x \in \mathrm{Re}$, there exists a unique additive inverse which we denote by $-x$ for which
  $$x + (-x) = (-x) + x = 0 \tag{43}$$
  For each $x \neq 0$ in Re, there exists a unique multiplactive inverse, which we denote by x^{-1} or $1/x$ for which:
  $$x \cdot \left(x^{-1}\right) = \left(x^{-1}\right) \cdot x = 1. \tag{44}$$

- It's not important to restrict the number of **definitions**, which are built from axioms, but it gets messy if we make more definitions that are really needed. (e.g. $4 \equiv 3 + 1$)

> **Definition**: **Positive integers** are the "natural numbers": $1, 2, 3, \ldots$ Note that:
> $$2 \equiv 1 + 1$$
> and so forth.

Definition: **Rational numbers** are in the form of:

$$\frac{a}{b} \equiv a \cdot \frac{1}{b}$$

where a, b, are integers are $b \neq 0$. Note that this uses axiom 5 with the definition of fractions to create rational numbers.

- There is no limit to the number of theorems. We can and should prove all *arithmetic* and *algebraic* theorems rigorously logically, starting from the Axioms (e.g. $4 = 2 + 2$).

Example 3: Let us prove $\sqrt{2}$ is irrational by contradiction. Suppose there is a pair of integers: a, b, such that:

$$\left(\frac{a}{b}\right)^2 = 2$$

where all common factors have been removed.Therefore:

$$\therefore a^2 = 2b^2 \tag{45}$$
$$\therefore a^2 \equiv 0 \pmod 2 \tag{46}$$
$$\therefore a \equiv 0 \pmod 2 \tag{47}$$
$$\tag{48}$$

We can write $a = 2q$ where q is some integer such that:

$$\therefore a^2 = 4q^2 \tag{49}$$
$$\therefore 4q^2 = 2b^2 \tag{50}$$
$$\therefore b^2 = 2q^2 \tag{51}$$
$$\therefore b^2 \equiv 0 \pmod 2 \tag{52}$$
$$\therefore b \equiv 0 \pmod 2 \tag{53}$$

However, since a and b are both even, we have contradicted our statement that all common factors have been removed. Thus $\sqrt{2}$ cannot be rational and can only be irrational.

- However, the 5th field axiom only discusses the creation of rational numbers. We could simply add a "root 2 axiom" to create $\sqrt{2}$, just like we did for 0 and 1.

- This is super messy because it would imply we'd need another axiom for every irrational, including every root of every **polynomial function**:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{54}$$

  where a can be any specified number and n is a positive integer. If we set $P_n(x) = 0$, we get a polynomial equation where we can find the roots.

Definition: If z is a root of $p_n(x)$ then $p_n(z) = 0$.

There are n roots[1], most are irrational and are called **algebraic numbers**.

- To prevent creating numerous new axioms, we create a new axiom called **CORA**: Completeness of the Reals Axiom, which tells us that every non-empty set of real numbers that is bounded above has a least upper bound among the real numbers.

[1] Per the fundamental theorem of algebra, which is beyond the scope of this course

Definition: A set of real numbers, $\mathbb{S}$ is bounded above if and only if there exists some number M such that $x \leq M$ for all $x \in \mathbb{S}$. For example:

$$\mathbb{S}_1 = \{1, 3, \frac{17}{5}, 211\} \tag{55}$$

Here, $M = 211$ or 250, etc. We can write the first upper bound as:

$$\text{ub}\mathbb{S}_1 = 211 \tag{56}$$

Definition: The least upper bound is the smallest of all the upper bounds. Here:

$$\text{lub}\mathbb{S}_1 = 211 \tag{57}$$

- Note that we **do not** require that $\text{lub}\mathbb{S} \in \mathbb{S}$ necessarily. For example, if:

$$\mathbb{S}_2 = \{x : x^2 < 2\} \tag{58}$$

  There are several upper bounds in this set, but is there a least upper bound? We may think intuitively it is $\sqrt{2}$, but we have to be careful! We haven't proved it exists yet. **However,** CORA has *creates* this new number, $\sqrt{2}$, by demanding that it exists.[2]

- Additionally, CORA does the same for all **algebraic irrationals** and **transcendental irrationals**. Without CORA, there would be no irrational numbers!

[2]However, proving that the lower upper bound is $\sqrt{2}$ is rather tricky and was removed from the supplement.

3 Review of Inequalities

- The absolute value is defined as

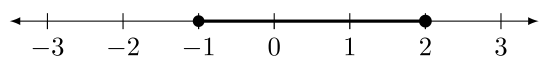
$$|a| = \begin{cases} +a & a \geq 0 \\ -a, & a < 0 \end{cases} \tag{59}$$

Note that $|a| \geq 0$ always.

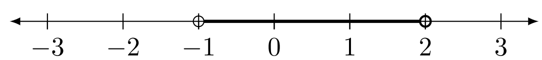
> **Warning**: Note that $|a| = \sqrt{a^2}$. This means that the square root is a function that only has one answer. The square root is defined such that $\sqrt{a} \geq 0$ if it exists and does not exist if $a < 0$. This is **different** from solving the equation:
>
> $$x^2 = 4 \tag{60}$$
>
> We want to perform the inverse of a square, which is *not* the square root! Instead, the inverse of x^2 is $\pm\sqrt{x}$. Just because there are two different values when squared gives the same number, doesn't mean that taking the square root of this number will yield two answers!

- The **real number line** is a geometric analogue of real numbers. It is not necessary (for rigorous proofs), but useful.

- A closed interval is represented by $[a, b]$, and can be written as $a \leq x \leq b$. For example, the interval $[-1, 2]$ can be represented by:

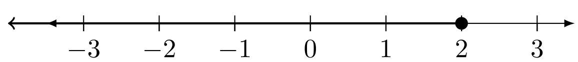


- An open interval does not contain the endpoints. It is represented by (a, b) or $a < x < b$. Similarly, this can be represented on a number line:



- A half-closed or half-open interval is when only one of the ends are closed, and is denoted by $[a, b)$ or $(a, b]$.

- If an interval goes to infinity, such as $(-\infty, b]$ which is equivalent to $x \leq b$. Note that this does not imply infinity is a number (or else the interval could be closed), but instead we can define an expression where infinity is embedded into such that it is rigorously logical.



- There are two sets of numbers involved in **functions**: an x-set and a y-set.

> **Definition**: A function is any **rule** that assigns each x-number to *one* y-number.

Note that any prescription for the function is acceptable. For example, a table is an example of a function. Neither the x's or y's have to include numbers. There can be holes!

- For each x, only a single value of y can be assigned, i.e. double functions are not allowed.[3]

- Usually we specify functions using an algebraic equation. If for some set of x's, the equation gives a real number y, then it is *assumed* that this *specifies* the set of x's for this function. For example:

$$f(x) = \begin{cases} \dfrac{10x + 5x^2}{x} = 10 + 5x & x \neq 0 \\ \text{DNE} & x = 0 \end{cases} \tag{61}$$

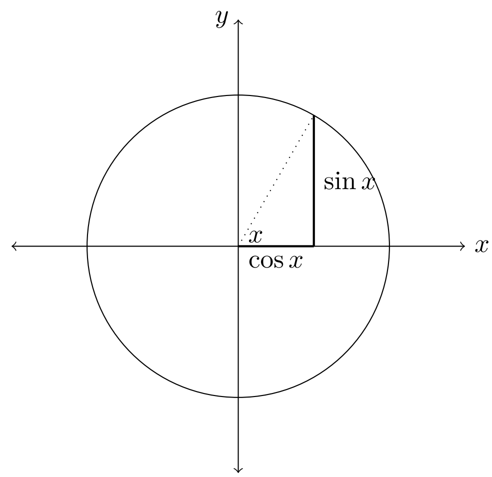
This is a perfectly good function since it assigns each number in the x-set for this function to some y-value.

[3]This is only from how we are defining functions. Other definitions may allow double valued functions.

- The **domain of a function** is the set of x-values and the **range of a function** as the set of y-values. Here, x is the independent variable and y is the dependent variable.

  Note that this doesn't work the other way around, since each x can be used to specify one y but each y can have several corresponding x values! There is some asymmetry involved.

- We want to define trigonometric functions purely algebraically, but for now we have to do it geometrically.



  where the angle is in radians. Radians are picked as the unit such that the arc (curved part subtended by angle of θ) is given by

$$s = Rx \tag{62}$$

  where R is the radius of the circle.

- Not all algebraic expressions are functions. Suppose we have an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1 \tag{63}$$

  and solving for y gives:

$$y = \pm b \left(1 - \frac{x^2}{a^2}\right)^{1/2} \tag{64}$$

  Since we can't have double valued functions, we can instead break this up into *two separate functions*

- Composite functions can be written in the form of $f(g(x))$.

> **Example 4:** Let $f(x) = x^2 + 2$ and $g(x) = \sin x$. Then:
>
> $$f(g(x)) = \sin^2(x) + 2 \tag{65}$$
> $$g(f(x)) = \sin\left(x^2 + 2\right) \tag{66}$$
>
> must be true.

- We also need a rigorous way to define **increasing** and **decreasing functions**.

> **Definition**: $f(x)$ is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.[a]
>
> [a]The motivation behind this definition is to forego the ambiguity when we try to say something like "As x gets bigger, y gets bigger." However a function is a definite relationship between pairs of numbers, none of the values are actually changing!

  Similarly, we can define a decreasing function to be the converse, where $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.[4]

[4]Note that we can't define an increasing or decreasing function based on its derivative since $y = x^3$ is increasing but it has a derivative of zero at $x = 0$.

Example 5: Prove that $f(x) = x^2 + 3$ is increasing for $x > 0$.

Consider any two numbers x_1, x_2 such that

$$0 < x_1 < x_2. \tag{67}$$

Multiplying by x_1, we get:

$$0 < x_1^2 < \boxed{x_1 x_2}. \tag{68}$$

We can also multiply by x_2, we get:

$$0 < \boxed{x_1 x_2} < x_2^2 \tag{69}$$

Comparing these two inequalities by comparing the boxed compressions, we show that:

$$x_1^2 < x_2^2 \tag{70}$$

$$x_1^2 + 3 < x_2 + 3 \tag{71}$$

$$f(x_1) < f(x_2) \tag{72}$$

Therefore, $f(x)$ is increasing on the interval $x > 0$.

Example 6: Prove that $f(x) = x^2 + 3$ is decreasing for $x < 0$.

Consider any two numbers x_1, x_2 such that

$$x_1 < x_2 < 0. \tag{73}$$

Multiplying by x_1, we get:

$$x_1^2 > \boxed{x_1 x_2} > 0 \tag{74}$$

since we are multiplying by a negative number. We can also multiply by x_2 (which is also negative) to get:

$$\boxed{x_1 x_2} > x_2^2 > 0 \tag{75}$$

Comparing these two inequalities by comparing the boxed expressions, we show that:

$$x_1^2 > x_2^2 \tag{76}$$

$$x_1^2 + 3 > x_2 + 3 \tag{77}$$

$$f(x_1) > f(x_2) \tag{78}$$

Therefore, $f(x)$ is decreasing on the interval $x < 0$.

- We can also define odd and even functions:

Definition: A function $f(x)$ is even if $f(-x) = f(x)$ for all x in the domain of $f(x)$. Similarly, $f(x)$ is odd if $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

- For arithmetic equalities and inequalities, we are given *arithmetic statements*:
  - Equality (e.g. $1 + 2 = 6 - 3$)
  - Inequality (e.g. $3 < 5$)

Theorem: You can add, subtract, multiply, or divide both sides by the same factor (positive or negative) and get another true statement.

However, for *inequalities* with a *negative factor* and for *multiplication and division* only, one has to *reverse the sign* of the inequality.[a]

[a] To convince yourself why this is true, suppose we start with $3 < 5$ and multiply both sides by -7. Is it true that $-21 < -35$?

4 Motivation Behind Delta-Epsilon

- An arithmetic equation or statement can be true or false. An *algebraic* equation (e.g. $x+3=7x-1$) is neither true or false. Instead it is used to *specify* a value of x (i.e. a *number*)
- An arithmetic inequality is either true or false. However, an algebraic inequality specifies a set of x. AS a result, we need to make a clear distinction between an arithmetic statement with an algebraic equality or inequality.
- Algebraic inequalities follow the same rules as arithmetic inequalities as summarized in lecture 3.

Example 7: Multiply both sides of $x < -4$ by -3. Do we get the same set of $x's$

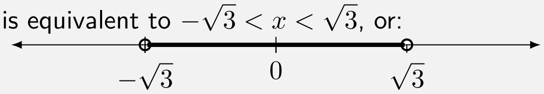
$$(-3)(x) > (-3)(4) \tag{79}$$

$$-3x > -12 \tag{80}$$

$$-x < 4 \tag{81}$$

For more practice, refer to the example in Lecture 3 for a discussion on how to prove x^2+3 decreases for $x < 0$.

Example 8: Show that $x^2 < 3$ is equivalent to $-\sqrt{3} < x < \sqrt{3}$, or:



Note that $x^2 \geq 0$ for all x. $\therefore \sqrt{x^2} < \sqrt{3}$. This is exactly equivalent to defining $|x| = \sqrt{x^2} < \sqrt{3}$. Therefore, there are two possibilities. For $x \geq 0$:

$$x \geq 0 \tag{82}$$

$$\sqrt{x^2} = x \tag{83}$$

$$x < \sqrt{3} \tag{84}$$

$$0 \leq x < \sqrt{3} \tag{85}$$

and similarly for $x < 0$, we have:

$$x < 0 \tag{86}$$

$$\sqrt{x^2} = -x \tag{87}$$

$$-x < \sqrt{3} \tag{88}$$

$$-\sqrt{3} < x < 0 \tag{89}$$

Combining, we get:

$$-\sqrt{3} < x < \sqrt{3} \tag{90}$$

Example 9: What set of x's is represented by $5(x^2-x-6) > 0$?

Note that the 5 has no effect. Factoring, we get:

$$(x-3)(x+2) > 0 \tag{91}$$

We **break it up** into different cases. First, both factors can be positive. This means that:

$$x-3 > 0 \tag{92}$$

$$\text{and } x+2 > 0 \tag{93}$$

$$\therefore x > 3 > -2 \tag{94}$$

which gives $x > 3$. Second, both factors can be negative. This means that:

$$x - 3 < 0 \quad (95)$$
$$\text{and } x + 2 < 0 \quad (96)$$
$$\therefore x < -2 < 3 \quad (97)$$

which gives $xM \leq 2$. Therefore, the set of x's that satisfy this inequality is:

$$x \in (-\infty, -2) \cup (3, \infty) \quad (98)$$

We should also perform some checks, such as picking numbers in the range $x < -2$, $-2 \leq x \leq 3$, and $x > 3$ to see if they match up with our solution.

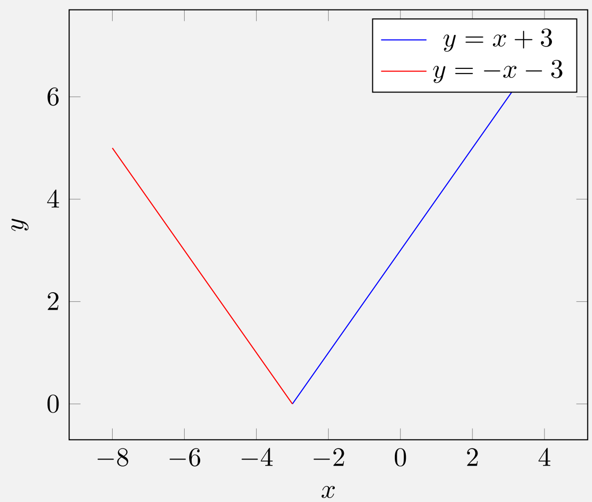
- Similarly, we need a systematic method to approach absolute value functions.

Example 10: What does $f(x) = |x + 3|$ look like?

Intuitively, this should look like the absolute value function $|x|$ but shifted three to the left. We can show this rigorously and algebraically by writing

$$f(x) = \begin{cases} x + 3, & \text{if } (x + 3) \geq 0 \implies x \geq -3 \\ -(x + 3), & \text{if } (x + 3) < 0 \implies x < -3 \end{cases} \quad (99)$$

which we can plot below as:



Example 11: What values of x satisfy $|x + 3| = 5$?

There are two possibilities. First,

$$(x + 3) \geq 0 \implies x \geq -3 \quad (100)$$

Therefore:

$$x + 3 = 5 \implies x = 2 \quad (101)$$

which satisfies the initial restriction posed. The second possibility is when the expression inside the absolute value function is negative:

$$(x + 3) < 0 \implies x < -3 \quad (102)$$

Therefore:

$$-x - 3 = 5 \implies x = -8 \quad (103)$$

which satisfies the $x < -3$ condition. As a result, both $x = 2$ and $x = -8$ satisfy this equality.

Example 12: What values of x satisfy $|x + 3| < 5$?

There are two possibilities. First,

$$(x + 3) \geq 0 \implies x \geq -3 \tag{104}$$

Therefore:

$$x + 3 < 5 \implies x < 2 \tag{105}$$

so we have $-3 \leq x < 2$. The second possibility is when the expression inside the absolute value function is negative:

$$(x + 3) < 0 \implies x < -3 \tag{106}$$

Therefore:

$$-x - 3 < 5 \implies x > -8 \tag{107}$$

so we can also have $-8 < x < -3$. Combining them both together, we get:

$$-8 < x < 2 \tag{108}$$

Idea: Note that the above example represents a *band* of x's centered on -3 of half-width 5. This means that an inequality such as:

$$|x - 7| < 2 \tag{109}$$

represents a band centered on 7 of half-width 2.

This leads to the introduction of $\delta - \epsilon$ proofs, where we want the restriction

$$|x - c| < \delta \tag{110}$$

where c, δ are given numbers. c may be negative or positive but $\delta > 0$ is always positive. This gives the set of x's that satisfy:

$$c - \delta < x < c + \delta \tag{111}$$

We will soon use this in defining the limit where it will be important to exclude the center point $x = c$. We can do this by rewriting the inequality as:

$$0 < |x - c| < \delta \tag{112}$$

forbidding the $x = c$ case. Similarly, we can write:

$$|f(x) - L| < \epsilon \tag{113}$$

to denote the fact that the y value is "sandwiched" in between $c - L$ and $c + L$.

5 Delta-Epsilon Examples

- We want to rigorously create the rigorous definition of a limit. For example, how does one show that for a function such as equation (12), that the limit:

$$\lim_{x \to 0} f(x) \tag{114}$$

does exist.

> **Idea**: We need to build a rigorous "test-definition" for the new number $\lim_{x \to c} f(x)$. We need to be given:
>
> 1. c, some particular x value
> 2. $f(x)$ which may not exist *at* c, but $f(x)$ *is* defined for **all** x *near* c.
> 3. A guess or candidate for $\lim_{x \to c} f(x)$ which we call L
>
> It is then imposed on you some positive number $\epsilon > 0$, which may be extremely small but never zero. Note that we are not told this exact value for ϵ and will have to allow for *any* $\epsilon > 0$.
>
> The challenge-test is: "Can you find some number $\delta > 0$[a], such that for all x's in the $x - band$, (i.e. in the set $0 < |x - c| < \delta$) the corresponding values of f fall somewhere inside the y-band, i.e. the set $|f(x) - L| < \epsilon$.
>
> Note that only δ is under your control.
>
> [a] Here δoggy δoggy!

6 Infinite Limits

- We continue to define the definition of the limit.

> **Definition**: If for any $\epsilon > 0$, a $\delta > 0$ can be found such that for all $0 < |x - c| < \delta$, it can be proved that $|f(x) - L| < \epsilon$, then we can define $\lim_{x \to c} f(x)$ to be a real number and we can assign it to the value L.

- A useful tool is the expression $x = \min\{a, b\}$. For example, for every, for every member you introduce to your club, you get $10, up to a maximum of $50. The expression is then $x = \min\{10N, 50\}$ where N is the number of new numbers.

Example 13: Prove that $\lim_{x \to 5} x^2 = 25$.

1. $\epsilon > 0$ is specified.
2. It is required that $|f(x) - L| < \epsilon$ or
$$|x^2 - 25| < \epsilon \tag{115}$$
3. when $0 < |x - c| < \delta$ or:
$$0 < |x - 5| < \delta \tag{116}$$
4. The left hand side of (2) becomes:
$$\begin{aligned} LHS &= |x^2 - 25| &(117)\\ &= |(x-5)(x+5)| &(118)\\ &= |x-5||x+5| \leq \delta|x+5| &(119) \end{aligned}$$
where we have applied the basic theorem of algebra in the last step. We now need to specify a *second* feature of δ, additional to anything we will specify in ste (5), i.e. in terms of ϵ. Here, we can specify a guess: $\delta \leq 1$. From (3),
$$\begin{aligned} |x-5| < \delta \leq 1 &&(120)\\ 5-1 \leq x \leq 5+1 &&(121)\\ 4 \leq x \leq 6 &&(122)\\ 9 \leq x+5 \leq 11 &&(123) \end{aligned}$$
Note that $x + 5 \geq 9$, then $x + 5 > 0$. As a result, it is positive and:
$$|x+5| = (x+5) \tag{124}$$
Note also that $x + 5 \leq 11$. This is helpful when comparing it to equation (119). Therefore:
$$LHS \leq \delta|x+5| \leq 11\delta \tag{125}$$
5. We now need to pick δ in terms of ϵ. We can try:
$$\delta = \frac{\epsilon}{11} \tag{126}$$
then plugging it into :
$$LHS < 11 \cdot \frac{\epsilon}{11} = \epsilon \tag{127}$$
However, we don't forget our override condition:
$$\delta = \min\{\epsilon/11, 1\} \tag{128}$$

- We can similarly define left and right hand limits. The left hand limit can be written as:
$$\lim_{x \to 0^-} f(x) \tag{129}$$
and similarly for the right hand limit
$$\lim_{x \to 0^+} f(x) \tag{130}$$

- For a right hand limit:

> **Definition**: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that for all $c < x < c + \delta$, one can prove $|f(x) - L| < \epsilon$, then $\lim_{x \to c^+} f(x) = L$

and we can similarly define it for the left hand limit.

- Note that:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} = \lim_{x \to c^-} f(x) = L \tag{131}$$

Example 14: Prove that $\lim_{x \to 0^+} x^{1/2} = 0$.

- $\epsilon > 0$ is specified.
- It is required that:

$$|\sqrt{x} - 0| < \epsilon \tag{132}$$

$$|\sqrt{x}| < \epsilon \tag{133}$$

$$\sqrt{x} < \epsilon \tag{134}$$

- when $0 < x < \delta$.
- From (2) and (3), we have:

$$x^{1/2} < \delta^{1/2} \tag{135}$$

under δ control!
- Try $\delta = \epsilon^2$. Then:

$$|\sqrt{x} - 0| < \delta^{1/2} = \epsilon \tag{136}$$

and we are done. We can also write this compactly.

Given $\epsilon > 0$, choose $\delta = \epsilon^2$, then when $0 < x < \delta$, $|\sqrt{x} - 0| < \epsilon$, therefore:

$$\lim_{x \to 0^+} \sqrt{x} = 0. \tag{137}$$

- We can also deal with infinite limits, such as:

$$\lim_{x \to 0} \frac{1}{x^4} \tag{138}$$

We can approach this rigorously: Imagine your ene$M$y imposes some very large number $M > 0$, say $M = 10^6$. The challenge then becomes: "Can you find a $\delta > 0$ such that for all $0 < |x - 0| < \delta$, such that $f(x) > M$?" If yes, we can write:

$$\lim_{x \to 0} \frac{1}{x^4} = \infty \tag{139}$$

> **Warning**: Note that this is **not** an equation as ∞ is **not** a number! All this does is a compact way of saying "$f(x)$ increases without limit as x approaches 0."

7 More on Infinite Limits

- Suppose we wish to rigorously prove an infinite limit.

Example 15: Prove $\lim_{x \to 0} \frac{1}{x^4} = \infty$.

1. Given: $M > 0$
2. It is required that $f(x) > M \implies \frac{1}{x^4} > M$
3. when $0 < |x - c| < \delta \implies 0 < |x| < \delta$
4. LHS of 2 is $|x| < \delta \implies \frac{1}{\delta^4} < \frac{1}{|x|^4}$. Therefore, the LHS of (2) is under δ control.
5. Judgement: Try $\delta = \frac{1}{M^{1/4}}$. Therefore:

$$\text{LHS}(2) > \frac{1}{\delta^4} = M \tag{140}$$

Given $M > 0$, choose $\delta = \frac{1}{M^{1/4}}$ then when $0 < |x - 0| < \delta$, we have $\frac{1}{x^4} > M$.

- Instead of having to prove $\lim_{x \to c} f(x) = L$. What if we were given this statement? This leads to limit theorems:

Theorem: Given $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then:

$$\lim_{x \to c} [f(x) + g(x)] = L + M \tag{141}$$

Proof:

1. Given: $\epsilon > 0$ is specified.
2. It is required that:

$$|f(x) + g(x) - L - M| < \epsilon \tag{142}$$

3. ... when $0 < |x - c| < \delta$
4. The left hand side of (2) is:

$$|(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| \tag{143}$$

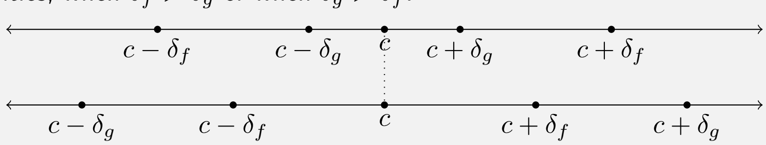
   where we have applied the triangle inequality. Note that $\lim_{x \to c} f(x) = L$ is given here. As a result, we can now play the role of the ϵnemy and we can specify an number $\epsilon_f > 0$ we want. Suppose we choose $\epsilon_f = \frac{\epsilon}{2}$. It is then guaranteed that some number $\delta_f > 0$ exists for sure such that for all $0 < |x - c| < \delta_f$ it can be proved that $|f(x) - L| < \epsilon_f = \frac{\epsilon}{2}$.

   Similarly for $g(x)$ we can impose any $\epsilon_g > 0$ we want, say $\epsilon_g = \epsilon/2$, and it is guaranteed that some $\delta_g > 0$ exists such that for all $0 < |x - c| < \delta_g$, then $|g(x) - M| < \epsilon_g = \epsilon/2$. Next, note that if x is inside both δ_f and δ_g bands then:

$$\text{LHS (2)} < \epsilon/2 + \epsilon/2 = \epsilon \tag{144}$$

   as requested.
5. Therefore, pick $\delta = \min\{\delta_f, \delta_g\}$.

There are two possibilities, when $\delta_f > \delta_g$ or when $\delta_g > \delta_f$:



so when $0 < |x - c| < \delta = \min\{\delta_g, \delta_f\}$, then x satisfies both $0 < |x - c| < \delta_f$ and $0 < |x - c| < \delta_g$.

Theorem: The product limit theorem says that ${}_{x\to c}f(x)g(x) = LM$ provided that both limits exist:

$$\lim_{x\to c} f(x) = L,\ \lim_{x\to c} g(x) = M \tag{145}$$

Do not use this theorem unless both limits exist!

Theorem: The polynomial limit theorem says that $\lim_{x\to c} P_n(x) = P_n(c)$ given that $P_n(x)$ is a polynomial.

Theorem: The rational function limit theorem:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{1}{M},\ M \neq 0 \tag{146}$$

and only applies when both limits exist.

Theorem: The root limit theorem says that:

$$\lim_{x\to c} f(x)^{1/n} = L^{1/n} \tag{147}$$

All of these limit theorems can be proven the same way as the additivity limit theorem, but proofs are not assigned. Therefore, the following is a completely rigorous proof.

Example 16: Determine $\lim_{x\to -2} \dfrac{x^2 - x - 6}{x^2 - 4}$.

We can write it as:

$$\lim_{x\to -2} \frac{x^2 - x - 6}{x^2 - 4} = \lim_{x\to -2} \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} \quad \text{(ok since } x+2 \neq 0 \text{ here.)} \tag{148}$$

$$= \frac{\lim_{x\to -2}(x-3)}{\lim_{x\to -2}(x-2)} \quad \text{(rational function LT)} \tag{149}$$

$$= \frac{-2-3}{-2-2} \quad \text{(polynomial LT)} \tag{150}$$

$$= \frac{5}{4} \tag{151}$$

- Another useful limit theorem is the sandwich limit theorem.

Theorem: Given:
  - $\lim_{x\to c} f(x) = L$
  - $\lim_{x\to c} h(x) = L$
  - $f(x) \leq g(x) \leq h(x)$ near c, but not necessarily *at* c.

Then:

$$\lim_{x\to c} g(x) = L \tag{152}$$

Example 17: Determine $\lim_{x\to 0} x^2 \cos^2\left(\dfrac{1}{x^2}\right)$

We might naively try to apply the product LT, but $\lim_{x\to 0} \cos^2\left(\frac{1}{x^2}\right)$ is not defined! Instead, we can apply the sandwich LT. Note that:

$$0 \le \cos^2\left(\frac{1}{x^2}\right) \le 1 \tag{153}$$

provided that $x \neq 0$. We can multiply both sides by x^2 since it is a positive quantity, then:

$$0 \le x^2 \cos^2\left(\frac{1}{x^2}\right) \le x^2 \tag{154}$$

We can rigorously find the limits of the two extremes of the inequality. We can define:

$$f(x) \equiv 0 \tag{155}$$

$$g(x) \equiv x^2 \cos^2\left(\frac{1}{x^2}\right) \tag{156}$$

$$h(x) \equiv x^2 \tag{157}$$

Note that:

$$\lim_{x\to 0} f(x) = 0 \tag{158}$$

$$\lim_{x\to 0} h(x) = 0 \tag{159}$$

so by the sandwhich limit theorem:

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x^2}\right) = 0. \tag{160}$$

8 Continuity

- A "continuous function" is intuitively clear, but how do we define it rigorously?

Definition: $f(x)$ is "continuous at c" if

$$\lim_{x\to c} f(x) = f(c) \tag{161}$$

Definition: A function $f(x)$ is discontinuous at c if it is not continuous.

- There are various types of discontinuity:
  - **Jump Discontinuity**- For example:

$$f(x) = \frac{|x|}{x} \tag{162}$$

  - **Removable Discontinuity**- For example:
  - Discontinuity because either $f(c)$ DNE or $\lim_{x\to c} f(x)$ DNE, or both.
- There are also continuity theorems.

Theorem: If $f(x)$ is continuous at every $x \in [a, b]$, then $f(x)$ is integrable on $[a, b]$ i.e. $\int_a^b f(x)\,\mathrm{d}x$ exists.

Theorem: Given f, g is continuous at a, then $f(x) + g(x)$ is continuous at a.

Proof: Apply the additivity L.T lmao.

- There is also such a thing as a one-sided continuity.

Definition: $f(x)$ is continuous on the right at c if $\lim_{x\to c^+} f(x) = f(c)$.

- amd for an interval.

Definition: $f(x)$ is continuous on (a, b) iff $f(x)$ is continuous at all $x \in (a, b)$.

Definition: $f(x)$ is continuous on $[a, b]$ iff $f(x)$ is continuous on (a, b) and $f(x)$ is continuous from the right of a and from the left of b.

Theorem: If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

- We can now introduce the **Intermediate Value Theorem**. Recap: We simply want there to be a number q such that $q \cdot q = 2$, i.e. $\sqrt{2}$ exists. We proved such a number does not exist among the rationals. So we imposed a new axiom CORA and then defined:

$$\sqrt{2} = \text{lub}\{x : x^2 < 2\} \tag{163}$$

which CORA guarantees exists as a real number. This however doesn't tell us that $\sqrt{2} \cdot \sqrt{2} = 2$, but we want to use this result. How can we rigorously prove that:

$$[\text{lub}\{x : x^2 < 2\}] \cdot [\text{lub}\{x : x^2 < 2\}] = 2 \quad (164)$$

Theorem:

1. Given that $f(x)$ is continuous on $[a, b]$
2. C is some number such that $f(a) < G(a) < f(b)$.
3. There exists some C in $[a, b]$ such that $f(C) = G$.

- The point of the IVT is that continuous functions don't skip over any y values. Note that this is a property that intuitively we want continuous functions to have.

Example 18: Prove that there is a number c such that $c \cdot c = 2$ using IVT rather than CORA directly.

Consider $f(x) = x^2$ on $[1, 2]$. It is easy to show that $f(x)$ is continuous on $[1, 2]$ per the polynomial continuous theorem. Here $f(1) = 1$ and $f(2) = 4$. Note that $1 < 2 < 4$ so $f(1) < 2 < f(2)$.

The IVT shows that there must be some number c where $1 < c < 2$ in this interval such that $f(c) = c \cdot c = 2$.

Note that if reals consisted of rationals only, then the IVT would not be true!

- Logically we could have started with the IVA (Intermediate Value Axiom) and used it to prove CORT (Completeness of Reals Theorem). But this would be messy: we would need to define functions before we had finished defining numbers.

Example 19: Prove there's at least one root of $x^{17} + 1 = 3x$ in $[0, 1]$.

Define $f(x) = x^{17} + 1 - 3x$. $f(x)$ is continuous by polynomial continuity theorem. $f(0) = 1$ and $f(1) = -1$, so $f(0) > 0 > f(1)$. By IVT there is some c in $(0, 1)$ such that $f(c) = 0$. Therefore c is a root of this equation.

9 Formal Definition of the Derivative

- We want to define a new number $f'(a)$ where it is defined as:

$$f'(a) \equiv \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{165}$$

if it exists where $a \in$ domain of $f(x)$. Note this is known as the derivative of $f(x)$ evaluated at $x = a$.

Example 20: Determine the derivative of $f(x) = x^3$ at $a - 1$. We can evaluate:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \tag{166}$$

$$= \lim_{h \to 0} \frac{(1+h)^3 - 1}{h} \tag{167}$$

$$= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \tag{168}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h} \tag{169}$$

$$= \lim_{h \to 0} (3 + 3h + h^2) \qquad \text{provided that } h \neq 0 \tag{170}$$

$$= 3 \qquad \text{polynomial limit theorem} \tag{171}$$

- h disappears when evaluating the limit, as a result, it is known as a **dummy variable**.
- The tangent line to the curve $f(x)$ at $x = a$ is given by the equation:

$$y_{\text{tangent line}}(x) \equiv f(a) + f'(a)(x - a) \tag{172}$$

- We can define average speed between t_0 and $t_0 + h$ to be:

$$v_{\text{avg}} = \frac{s(t_0 + h) - s(t_0)}{h} \tag{173}$$

where $s(t)$ is the displacement. We can use our rigorous definition of the derivative to make a rigorous definition of "speed at an instant t_0" to be:

$$v(t_0) \equiv \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} \tag{174}$$

- We can extend further and define a new *function* $f'(x)$ such that:

$$f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{175}$$

There are two variables here, but it is still rigorously defined since h is still a dummy variable and it will still disappear after we evaluate the limit.

Definition: If $f'(a)$ exists we say $f(x)$ is differentiable at a.

Definition: If $f'(a)$ is differentiable at all $x \in$ domain of $f(x)$, we can say that $f(x)$ is a differentiable function.

Example 21: Let $f(x) = x^2$. Find $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \tag{176}$$

$$= \lim_{h \to 0} \frac{(x_2^2 h x_h^2 - x^2}{h} \tag{177}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h} \tag{178}$$

$$= \lim_{h \to 0} (2x + h) \tag{179}$$

where we canceled as $h \neq 0$. This is a polynomial function of h as far as $\lim\limits_{h \to 0}$ is concerned! Therefore, by the polynomial limit theorem, we get:

$$f'(x) = 2x \tag{180}$$

Example 22: Let $f(x) = x^n$. Prove that $f'(x) = nx^{n-1}$.

$$f'(x) = \lim_{h \to 0} \frac{(x+n)^n - x^n}{h} \tag{181}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \cdots + h^n - x^n}{h} \tag{182}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \cdots + h^n}{h} \tag{183}$$

$$= \lim_{h \to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \cdots + h^{n-1} \tag{184}$$

$$= nx^{n-1} \tag{185}$$

Later we will show that this is true for any real number, and not just for positive integers.

- Note that there are different notations. For example, the Leibniz notation gives us:

$$f'(x) \equiv \frac{df}{dx} \tag{186}$$

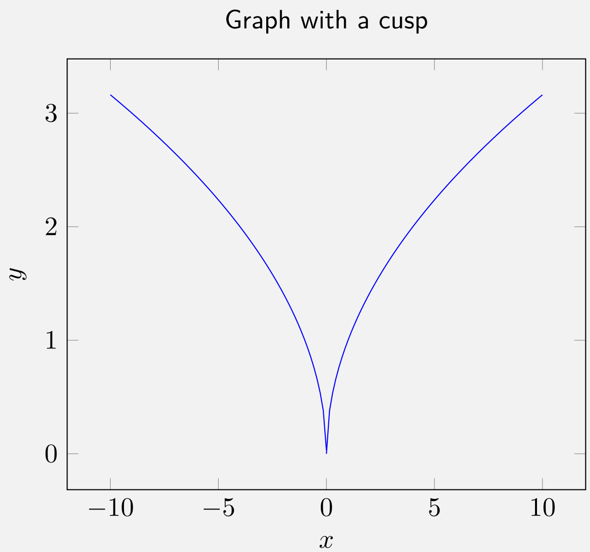
10 Differentiability and Derivative Theorems

- A few formal definitions:

Definition: $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at all $x \in (a, b)$.

Definition: $f(x)$ is differentiable on $[a, b]$ if:
  - $f(x)$ is differentiable on (a, b).
  - The right hand derivative at a exists.
  - The left hand derivative at b exists.

Example 23: There is a cusp in the following example:



Another example would be the absolute value of x, $|x|$.

Warning: From above example, $f(x)$ may be continuous at a point but no differentiable. Differentiability is rarer than continuity!

Theorem: Given $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

Proof: Consider

$$f(a+h) - f(a) = \left[\frac{f(a+h) - f(a)}{h}\right] h \tag{187}$$

which is acceptable if $h \neq 0$. Then:

$$\lim_{h\to 0}\{f(a+h) - f(a)\} = \lim_{h\to 0}\left[\frac{f(a+h) - f(a)}{h}\right] \cdot \lim_{h\to 0} h \qquad \text{(Product LT)} \tag{188}$$

$$= 0 \qquad \text{(Polynomial LT)} \tag{189}$$

Note that the use of the product limit theorem requires that both limits exist. We know the first limit exists since we are given that $f'(a)$ exists. As a result:

$$\lim_{h\to 0}\{f(a+h) - f(a)\} = 0 \tag{190}$$

$$\lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) = 0 \tag{191}$$

Note that per the polynomial limit theorem, we have $\lim_{h\to 0} f(a) = f(a)$. Making the substitution $x = a + h \implies h = x - a$, we can rewrite our expression as:

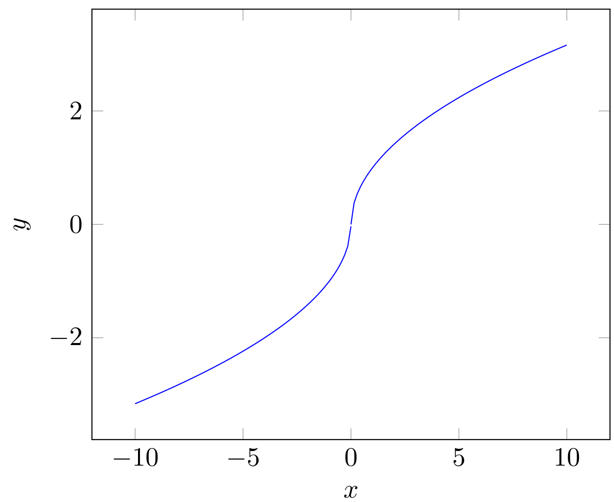
$$\lim_{h\to 0} f(a+h) = \lim_{x\to a} f(x) \tag{192}$$
$$\lim_{x\to a} f(x) = f(a) \tag{193}$$

and therefore $f(x)$ is continuous at $x = a$. Note that the reverse isn't necessarily true!

- **Vertical Tangent Lines** can exist. For example, if $f(x) = x^{1/3}$, then:

$$f'(x) = \frac{1}{3}x^{-2/3} \tag{194}$$

Vertical Tangent Line Example



Definition: A vertical tangent occurs when

$$\lim_{x\to c} |f'(x)| = \infty \tag{195}$$

and $f(x)$ is continuous at c.

- There are a few derivative theorems:

Theorem: The **constant derivative theorem**: For $f(x) = C$, then $f'(x) = 0$.

Theorem: Additivity D.T:

$$(f+g)' = f' + g' \tag{196}$$

which is true if both exist.

Theorem: The product D.T. is:

$$(fg)' = f'g + fg' \tag{197}$$

Theorem: The Power D.T: For $f(x) = Cx^n$, then $f'(x) = nCx^{n-1}$.

Theorem: The polynomial D.T. says that:

$$P_n'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 \tag{198}$$

Theorem: The reciprocal function D.T. says that:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2} \tag{199}$$

Proof: We can write

$$\left(\frac{1}{f}\right)' = \lim_{h\to 0}\left\{\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}\right\} \tag{200}$$

$$= \lim_{h\to 0}\left\{\frac{f(x) - f(x+h)}{hf(x)f(x+h)}\right\} \tag{201}$$

$$= \overbrace{\underbrace{\lim_{h\to 0}\left\{\frac{f(x) - f(x+h)}{h}\right\}}_{A} \cdot \underbrace{\lim_{h\to 0}\frac{1}{f(x)}}_{B} \cdot \underbrace{\lim_{h\to 0}\frac{1}{f(x+h)}}_{C}}^{\text{product limit theorem}} \tag{202}$$

Let us now deal with each limit individually. We have:

$$A = -f'(x) \tag{203}$$

per definition. For B, we can apply the constant limit theorem to get:

$$B = \frac{1}{f(x)} \tag{204}$$

To tackle C, because $\dfrac{1}{f(x)}$ is differentiable, it is continuous, so we can invoke the definition of continuity to get:

$$C = \frac{1}{f(x)} \tag{205}$$

and combining everything together:

$$\left(\frac{1}{f}\right)' = -\frac{f'(x)}{f(x)^2} \tag{206}$$

Example 24: If $f(x) = x^4$, what is $f'(x)$?

Set $g(x) \equiv x^4$, then $f(x) = \dfrac{1}{g(x)}$. Therefore:

$$f'(x) = -\frac{g'(x)}{g(x)^2} \qquad \text{reciprocal LT} \tag{207}$$

$$= \frac{-4x^3}{(x^4)^2} \tag{208}$$

$$= -4x^{-5} \tag{209}$$

So we have proved $(x^n)' = nx^{n-1}$ for even when n is a negative integer!

Theorem: The quotient derivative theorem says:

$$(f/g)' = \frac{f'g - fg'}{g^2} \tag{210}$$

- We can now tackle **rates of change**. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Therefore:

$$\frac{dV}{dr} \equiv V' = \frac{4}{3}\pi \underbrace{(3r^2)}_{P.D.T} = \underbrace{4\pi r^2}_{\text{surface area of the sphere!}} \tag{211}$$

Idea: Intuitively, this makes sense! We can interpret the derivative (in Leibniz notation) gives us that $\frac{dV}{dr}$ is a fraction. If we write it as small increments, then:

$$\underbrace{\Delta V}_{\text{small increment of volume}} \simeq 4\pi r^2 \underbrace{\Delta r}_{\text{small increment of radius}} \tag{212}$$

Note that this is only approximate. We can get the actual change in volume as:

$$\Delta V_{\text{actual}} = \frac{4}{3}\pi \left[(r + \Delta r)^3 - r^3\right] \tag{213}$$

$$= \Delta V_{\text{approx}} \left(1 + \frac{\Delta r}{r} + \underbrace{\frac{1}{3}\left(\frac{\Delta r}{r}\right)^2}_{\text{goes to zero}}\right) \tag{214}$$

Therefore As $\Delta r \to 0$, we then have:

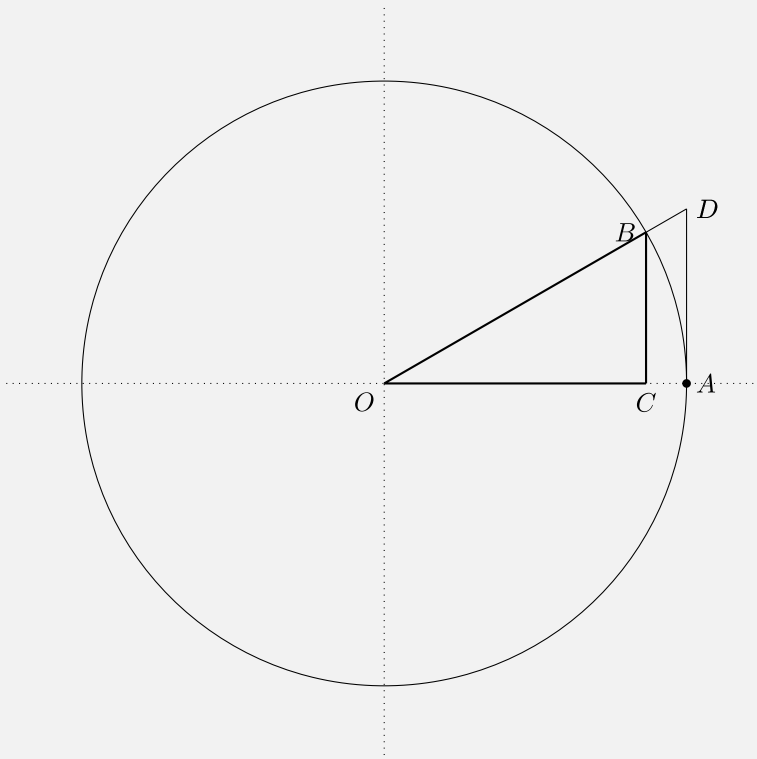
$$\Delta V_{\text{approx}} \to \Delta V_{\text{actual}} \tag{215}$$

11 Trig Functions / Derivatives

- We can deal with trigonometric functions.

Example 25: Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Note that we cannot use the product limit theorem and both do not exist at zero. We can do this geometrically by drawing a unit circle:



Let $x \equiv \angle BOC$. Then the area of $\triangle OBA$ is:

$$[\triangle OBA] = \frac{1}{2} \sin x \cdot 1 = \frac{1}{2} \sin x \tag{216}$$

The area of sector OBA is then:

$$[OBA] = \frac{1}{2} x \cdot 1^2 = \frac{1}{2} x \tag{217}$$

The area of $\triangle DOA$, using the fact that $DA = \tan x$ is:

$$[\triangle DOA] = \frac{1}{2} \tan x \cdot 1 = \frac{1}{2} \tan x \tag{218}$$

Therefore it is geometrically obvious that:

$$\sin x \leq x \leq \tan x \tag{219}$$

We can divide by two to get:

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \tag{220}$$

which is equivalent to:

$$\cos x \leq \frac{\sin x}{x} \leq 1 \tag{221}$$

We can then use the sandwich L.T. to prove that the limit is equal to one.

- We can find the derivative of sine functions:

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \tag{222}$$
$$= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \tag{223}$$
$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h} \tag{224}$$
$$= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} \tag{225}$$
$$= \sin x\cdot 0+\cos x\cdot 1 \tag{226}$$
$$= \cos x \tag{227}$$

Similarly we can show that:

$$\frac{d}{dx}\cos x = -\sin x \tag{228}$$

- For composite functions, we introduce the chain rule:

Theorem: The chain rule is given by:

$$f'(x) = f'(u)u'(x) \tag{229}$$

and is highly suggestive when written in Leibniz notation:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \tag{230}$$

Example 26: Suppose we have $f(x) = (3x^2+1)^{173}$. What is $f'(x)$?

$$u(x) \equiv 3x^2+1 \tag{231}$$

We can let $f(u) = u^{173} = f(u(x))$. Then:

$$\frac{du}{dx} = 6x \tag{232}$$
$$\frac{df}{du} = 173u^{172} \tag{233}$$

Therefore:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 6x(173u^{172}) \tag{234}$$
$$= 6x(173)(3x^2+1)^{172} \tag{235}$$

- Sometimes we only have $y(x)$ in the form of an implicit relationship, such as:

$$x^3y^7 - x^2 + y^2 = 0 \tag{236}$$

Note that we can't write $y(x)$ explicitly. While it is less convenient, we can relate y and x with a table and via numerical methods, but we can do it analytically as well. The trick is to apply the $\frac{d}{dx}$ operator to both sides of the implicit equation:

$$\frac{d}{dx}\left(x^3y^7 - x^2 + y\right) = \frac{d}{dx}0 \tag{237}$$
$$\frac{d}{dx}(x^3y^7) - \frac{d}{dx}x^2 + \frac{d}{dx}(y) = 0 \tag{238}$$
$$\tag{239}$$

The first term can be evaluated as:

$$\frac{d}{dx}(x^3y^7) = x^3\frac{d}{dx}y^7 + y^7\frac{d}{dx}x^3 \tag{240}$$

$$= 7x^3y^6\frac{dy}{dx} + 3x^2y^7 \tag{241}$$

After doing this for all terms, we get:

$$3x^2y^7 + 7x^3y^6y' - 2x + y' = 0 \tag{242}$$

and solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2x - 3x^2y^7}{7x^3y^6 + 1} \tag{243}$$

Proof: To prove that $\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{p/q-1}$, we can use the chain rule.

Let $u \equiv x^{p/q}$, and we want to find $u(x)$. Note that:

$$u^q = x^p \tag{244}$$

Define $f(u) \equiv u^q = x^p$. Therefore, $f(u(x))$ is a composite function. We can use the chain rule to get:

$$\frac{df}{dx} = px^{p-1} \tag{245}$$

$$\frac{df}{du} = qu^{q-1} \tag{246}$$

$$\tag{247}$$

We can divide the two to get:

$$u' = \frac{df}{dx}/\frac{df}{du} = \frac{px^{p-1}}{qu^{q-1}} \tag{248}$$

Recall that since $u = x^{p/q}$, we can rewrite:

$$u^{q-1} = x^{p-p/q} \tag{249}$$

therefore simplifying the derivative to:

$$u' = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} \tag{250}$$

$$= \frac{p}{q}x^{p/q-1} \tag{251}$$

Therefore we have proved that:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{252}$$

for any rational number n.

- We can also look at related rates now. For example, suppose we have the volume of a hailstone $V(t)$ and a radius $r(t)$ changing with time t. Suppose we are given that:

$$r(t) = 3t^2 + t \tag{253}$$

in appropriate units. To determine how fast V is changing at $t = 2$ min, then we can use the change rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \tag{254}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)(6t+1) \tag{255}$$

$$= (4\pi r^2)(6t+1) \tag{256}$$

Plugging everything in gives:

$$\left.\frac{dV}{dt}\right|_{t=2} = 20 \tag{257}$$

again in appropriate units since I'm too lazy to write them down.

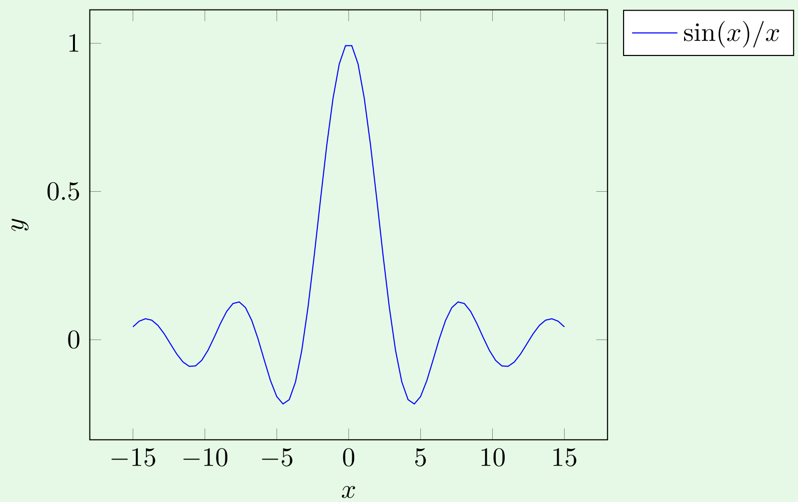
12 Applications of Derivatives

- Applications of Derivatives:

Definition: $f(x)$ has "an absolute maximum at c" if $f(c) \geq f(x)$ for all $x \in$ domain of $f(x)$. **Note** that $f(c)$ must exist!

For example, if $f(x) = \frac{\sin x}{x}$, it does not have an absolute maximum at $x = 0$!

Example



Definition: $f(x)$ has a "absolute max on $[a, b]$ etc" if $f(c) \geq f(x)$ for all $x \in [a, b]$

Definition: $f(x)$ has a "local max at c" if $f(c) \geq f(x)$ for *some* open interval containing c.

Theorem: The **extreme value theorem** (EVT) says that given $f(x)$ is continuous on $[a, b]$, then $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(x)$ for some $c, d \in [a, b]$.

However, functions do not need to be continuous to have an absolute max.

Proof. The outline of the proof is as follows:

1. Prove all continuous functions on $[a, b]$ are **bounded**
2. Then prove all continuous functions on $[a, b]$ have a max and a min.

Note that this is not the same thing! Remember that $f(x) = \frac{\sin x}{x}$ on $[-1, 1]$ is bounded, but does not have an absolute maximum! However, this doesn't violate it since it's not continuous.

We will take (1) to be proven and just prove (2): Consider the set $S = \{f(x) : a \leq x \leq b\}$. Since S is a set of f-values from (1), S is bounded above. By CORA, lub(S) exists as a real number ,call it M. Therefore: $f(x) \leq M$ for all $x \in [a, b]$

We now need to prove that there is some $c \ni [a, b]$ such that $f(c) = M$, i.e. $f(x)$ takes on the value M. We can prove this via contradiction:

Suppose $f(x)$ never equals M. We can then define:

$$g(x) \equiv \frac{1}{M - f(x)} \tag{258}$$

Note that $g(x) > 0$. (It cannot be negative since $M > f(x)$). Therefore, $g(x)$ is also continuous on $[a, b]$ by A.C.T, Q.C.T, and by the fact that $f(x) \neq M$.

Therefore $g(x)$ is also bounded above by part (1). There exists a number K such that $0 < g(x) \leq K$ where

$K > 0$. Taking the inverse, we have:

$$\frac{1}{K} \leq \frac{1}{g(x)} \tag{259}$$

$$\frac{1}{K} \leq M - f(x) \tag{260}$$

$$f(x) \leq M - \frac{1}{K} \tag{261}$$

This makes $M - \frac{1}{K}$ an upper bound of S. However, if M is the least upper bound of S, this gives a contradiction.

Since there is a contradiction, there exists at least one $c \in [a, b]$ such that $f(c) = M$. Therefore, a maximum exists. □

- Fermat's Theorem:

Definition: c is a "critical point" of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

- We have to be careful however, suppose we look at $f(x) = x^2$ on $x \in [1, 2]$. It is continuous by the polynomial C.T., and $f(2) = 4$ is an absolute maximum of $f(x)$ in $[1, 2]$. However, this doesn't violate Fermat's theorem since it's an absolute max, not a local max!

- Another example is $f(x) = x^3$. We have $f'(0) = 0$ but $f(0)$ is not a local max or min. We **cannot** reverse Fermat's theorem!

Idea: The motivation behind Fermat's theorem is as follows:

1. We often need to find local max, min.
2. But how can we?
3. It's usually easy to calculate $f'(x)$ and then find out where $f'(c) = 0$ or DNE.
4. While these critical points are not necessarily local max, min, only local max, min points will be in this set.[a]

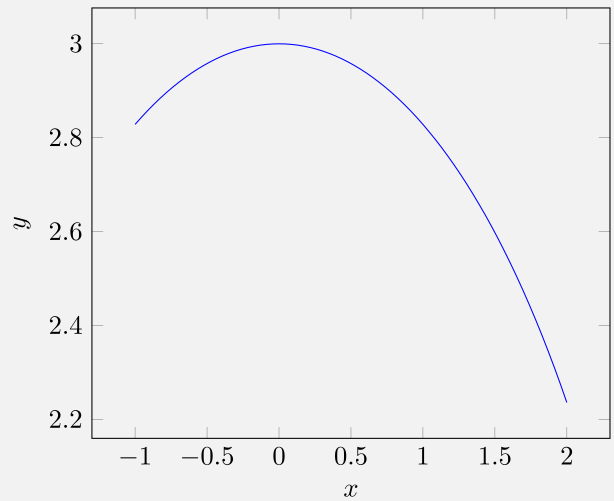
[a]Like using a net to catch fish. Not everything you catch will be the fish you want but the fish you want will be among it.

- This leads to a test for the absolute max/min on $[a, b]$. Given that $f(x)$ is continuous on $x \in [a, b]$. By the EVT there is an absolute max,min on $[a, b]$ for sure. Then, we can:

1. Find all c_{crit} and $f(c_{\text{crit}})$.
2. Find $f(a)$, $f(b)$.
3. The largest of these number is absolute max, and the smallest is the absolute min.

Example 27: Let $f(x) = (9 - x^2)^{1/2}$ on $x \in [-1, 2]$:

Example



We can find the derivative as:

$$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) \tag{262}$$

using the chain rule, polynomial D.T., power D.T.

1. We now look for when $f'(c)0 = 0$, which only happens when $f(0) = 3$. When is it undefined? One might be tempted to say at -3 or 3 but they aren't in the interval.
2. $f(-1) = \sqrt{8}$ and $f(2) = \sqrt{5}$
3. Therefore, the absolute maximum is $f(0) = 3$ and the absolute minimum is $f(2) = \sqrt{5}$.

13 The Mean Value Theorem

- We introduce the **Mean Value Theorem (MVT)**. First, we need to prove a simpler version known as **Rolle's Theorem**

Theorem: Given that f is continuous on $[a, b]$ and f is differentiable on (a, b) and $f(a) = f(b)$. Then there exists some $c \in (a, b)$ such that $f'(c) = 0$. Note that there may be more than one c

Proof: There are only three possibilities:
1. $f(x) > f(a)$ for some x in (a, b)
2. $f(x) < f(a)$ for some x in (a, b)
3. $f(x) = f(a) = f(b)$ for all x in (a, b)

To prove the third case, since $f'(x) = 0$ (C.D.T.), for all x in $[a, b]$, then it is automatically satisfied. To prove the first case, by the extreme value theorem, there is an absolute maximum in $[a, b]$. It can't be an end-point maximum so it by Fermat's principle it must be a critical point. It can't be a critical point where the derivative doesn't exist, so it must be a point where $f'(c_{\text{crit}}) = 0$. The second case is identical to the first.

Theorem: The **Mean Value Theorem:** Given that $f(x)$ is continuous on $[a, b]$ and $f(x)$ is differentiable on (a, b), then there exists some $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{263}$$

Note that both continuity and differentiability is needed.

Example 28: Physics example: Let $d(t)$ be the distance travelled in time t. The MVT tells us that at some time in the trip your instantaneous speed must equal your average speed on the trip.

Proof: The equation of a secant line is:

$$y_{\text{secant}}(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{264}$$

Note that this is in the form of $y_{\text{secant}}(x) = A + Bx$, a first order polynomial. Then we can define:

$$g(x) \equiv f(x) - y_{\text{secant}}(x) \tag{265}$$

We now show that $g(x)$ satisfies Rolle's theorem. $g(x)$ is continuous on $[a, b]$ since $f(x)$ is. Also $y_{\text{secant}}(x)$ is continuous (Poly CT) so $g(x)$ is also continuous (ACT). Similarly, $g(x)$ is also differentiable on (a, b) since $f(x)$ is. $y_{\text{secant}}(x)$ is also differentiable per Poly DT and $g(x)$ is also (ADT). We have $g(a) = g(b) = 0$, so by Rolle, there is some $c \in (a, b)$ such that $g'(c) = 0$ or:

$$f'(c) - y'_{\text{secant}}(c) = 0 \tag{266}$$

Using equation 264, we have:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{267}$$

Example 29: Given $f(x) = x^2$ on $[2, 3]$. Prove that f satisfies the conditions of MVT. We have $f(a) = 4$ and $f(b) = 9$ such that:

$$\frac{f(b) - f(a)}{b - a} = \frac{9 - 4}{3 - 2} = 5 \tag{268}$$

Is there some $c \in (2, 3)$ such that $f'(c) = 5$? Yes! We can let $f'(x) = 2x = 5 \implies x - 5/2$. Since $2 < 5/2 < 3$, then this all checks out.

General Case:

– Let $y = f(x)$ where y is the dependent variable and x is the independent variable.

– Define increments Δy and Δx, 2 *new* variables! They are related by:

$$\Delta y = f(x + \Delta x) - f(x) \tag{269}$$

Here Δx is the independent variable and Δy is the dependent variable.

Example Case:

– Let $y = x^{1/3}$. Choose $x = 27$ for example. We have free choice for picking the value of x.

– Choose $\Delta x = 2$, say for example. Remember we have free choice! Then:

$$\Delta y = 29^{1/3} - 27^{1/3} = 29^{1/3} - 3 \tag{270}$$

Idea: Note that the value of Δy depends on choices for both x and Δx!

– Define differentials dx, dy, 2 new variables related by:

$$dy \equiv f'(x)dx \tag{271}$$

– Choose $dx = 1/2$ for example. Then we can say:

$$dy = \underbrace{\frac{1}{3}(27)^{-2/3}}_{\text{derivative of f}}\left(\frac{1}{2}\right) = \frac{1}{54} \tag{272}$$

Idea: Since dx and Δx are both independent variables, we are free to choose them to be equal: $dx = \Delta x$. This can be useful! Note that this implies that:

$$\Delta y \approx dy \tag{273}$$

is an approximation, and this approximation improves as $\Delta x \to 0$. The practical point is that we may want to know Δy, i.e. $f(x + \Delta x) - f(x)$. It may, however might be easy to calculate $f'(x)$ such that:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x \tag{274}$$

Example 30: Suppose we wish to estimate $29^{1/3}$, then we can define $f(x) = x^{1/3}$ and pick $x = 27$, $\Delta x = 2$ such that:

$$\Delta y = f(x + \Delta x) - f(x) \tag{275}$$
$$= f(29) - f(27) = 29^{1/3} - 3 \tag{276}$$

We can now use the approximation $\Delta y \approx dy = f'(x)dx = f(x)\Delta x$ where:

$$\Delta y \approx \frac{1}{3}x^{-2/3}\Delta x = \frac{2}{27} \tag{277}$$

and:

$$\frac{2}{27} \approx 29^{1/3} - 3 \implies 29^{1/3} \approx 3 + \frac{2}{27} \tag{278}$$

14 Critical Points and Intervals of Increasing/Decreasing

- Using differentials, we estimated $29^{1/3} \approx 3.074$. We need to know how far it could be off.
- We can use the MVT to bracket our estimate. Apply the MVT to $f(x) = x^{1/3}$ on $[27, 29]$. There is some $c \in (27, 29)$ such that:

$$f'(c) = \frac{29^{1/3} - 27^{1/3}}{29 - 27} = \frac{29^{1/3} - 3}{2} \tag{279}$$

or:

$$f'(c) = \frac{1}{3}c^{-2/3} \tag{280}$$

Therefore:

$$29^{1/3} = 3 + \frac{2}{3}c^{-2/3} \tag{281}$$

The largest is when $c = 27 \implies c^{-2/3} = \frac{1}{9}$. The smallest is when $c = 29$, but we don't know what $29^{-2/3}$ is!

- Note however that $29 < 64$ such that:

$$29^{2/3} < 64^{2/3} = 16 \implies \frac{1}{16} < 29^{-2/3} \implies c^{-2/3} > \frac{1}{16} \tag{282}$$

and therefore:

$$3 + \frac{2}{3}\frac{1}{16} < 29^{1/3} < 3 + \frac{2}{3}\frac{1}{9} \implies 3.0416 < 29^{1/3} < 3.074 \tag{283}$$

- To graph functions, we need a few quick tests.
- QT1: First is the **Increasing/Decreasing Test**

> **Idea**: Given that f is differentiable on interval I, we show that:
> - If $f' > 0$, f is increasing.
> - If $f' < 0$, f is decreasing.
> - If $f' = 0$, f is constant.

We can prove the first statement.

Proof. Since f is differentiable, the MVT holds. There is some $c\ inI$ such that:

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \implies f(x_2) - f(x_1) \tag{284}$$

which is the definition of an increasing function. The proof goes similarly for the other two. □

- QT2: First derivative test. The motivation behind this is that $f(c_{\text{crit}})$ includes max and min values, but also others. How do we know which to keep?

> **Idea**: Given that I contains a critical point and f is continuous at c_{crit}. f is differentiable in I, but not necessarily at c_{crit}. Then:
> - If $f' > 0$ to the left of c_{crit} and $f' < 0$ is to the right, then c_{crit} is a local max.
> - If it's the opposite, we get the local minimum.

We can also prove this:

Proof. There is some a such that $f' > 0$ for $x \in (a, c)$, by QT1, f is increasing where:

$$f(c) \geq f(x) \tag{285}$$

in (a, c). There is also some b such that $f' < 0$ for $x \in (c, b)$ where $f' < 0$. By QT1, f decreases. As a result:

$$f(c) \geq f(x) \tag{286}$$

in (c, b). Therefore:

$$f(c) \geq f(x) \tag{287}$$

for all $x \in (a, b)$. Therefore, $f(c)$ is a local maximum, by definition. □

- Concavity: Points of Inflection

Definition: If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I.

- QT3: Concavity Test:

Idea: Given that $f(x)$ is twice differentiable on I, then $f''(x)$ exists on I. As a result:
  - If $f'' > 0$, f is concave up.
  - If $f'' < 0$, f is concave down.

Proof. Proof assigned (pg A43). Uses MVT and QT1. □

Definition: A point of inflection is at c if:
  - $f(x)$ is continuous at c and
  - Sign of concavity changes at c.

Example 31: Let $f(x) = x^3$. Then $f' = 3x^2$ and $f'' = 6x$. Since $f(x)$ is continuous at c and the sign of concavity changes at $x = 0$, therefore $(0, 0)$ is an inflection point.

- QT4: Second derivative test:

Idea: Given that $f''(x)$ is continuous near c and $f'(c) = 0$, then:
  - If $f''(c) > 0$, $f(c)$ is a local minimum.
  - If $f''(c) < 0$, $f(c)$ is a local maximum.
  - If $f''(c) = 0$, there is no verdict!

Note that this is even quicker than QT2!

Idea: In summary, the recipe to test for local max and min is to:
  - Find all c_{crit}.
  - If QT4 applies, use it.
  - If it doesn't, and if QT2 applies, use it.
  - If QT2 doesn't apply, use the basic definition of increasing/decreasing.

15 Defining Horizontal Asymptotes

- We can define horizontal asymptotes:

> **Definition**: A horizontal asymptote occurs when $\lim_{x\to\infty} f(x) = L$. We can say that $f(x)$ goes to L as x goes to infinity if for any $\epsilon > 0$, a number A can be found s.t. for all $x > A$, $|f(x) - L| < \epsilon$.
>
> The idea behind this revolves around finding f values as close to L as might be wanted by going to large enough x values.

- Geometrically, we can say that if $\lim_{x\to\infty} f(x) = L$, then the line $y = L$ is the horizontal asymptote of $f(x)$ at $x = \infty$.

> **Theorem**: The reciprocal horizontal asymptote limit:
>
> $$\lim_{x\to\pm\infty} \frac{1}{x^r} = 0 \tag{288}$$

- Slant asymptotes are a thing too. For example, suppose we have the function:

$$f(x) = \frac{x+2}{1+\frac{1}{x^2}} \tag{289}$$

We might say that intuitively $f(x) \to x + 2$ as $x \to \infty$.

> **Definition**: If $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$, then $y = mx + b$ is a slant asymptote to $f(x)$ at $+\infty$.

- Curve Sketching Check-list:
  - Find domain/range/limits at infinity, end points if they exist, vert/horz/slant asymptote
  - Intercepts: Find x/y intercepts.
  - Establish if $f(x)$ is symmetrical/even/odd/periodic
  - Find $f'(x)$ then find all critical points and $f(c_{\text{crit}})$. Find when $f(x)$ is increasing/decreasing. Use 1st derivative test (QT2). Find vertical tangents/cusps if they exist.
  - Find $f''(x)$, find where $f(x)$ is concave up/down. Find points of inflection if they exist. (Optional: use 2nd derivative test QT4 to confirm local max/min)
  - Choose largest and smallest values of f from the above as abs. max, min, if they exist.

16 Max/Min Problems

- When often setting up max min problems, we might have two variables.

Example 32: Suppose that the area of two shapes is $A = x^2 + \pi y^2$ and the total perimeter is a fixed 28 units. Then:

$$4x + 2\pi y = 28 \tag{290}$$

$$x = 7 - \frac{\pi}{2}y \tag{291}$$

$$A(y) = \left(7 - \frac{\pi}{2}y\right)^2 + \pi y^2 \tag{292}$$

We can then find the domain of $A(y)$: $0 \le y \le \frac{14}{\pi}$. Endpoints are important! We can have a checklist:

- Check critical points. When is $A' = 0$? Occurs when $y_{\text{crit}} = \frac{7}{2 + \pi/2} \approx 2$ cm.
- Check for Endpoints
- Check for local max, min
- Check $\lim\limits_{y \to \infty}$. Doesn't apply here.
- Decision. help that midterm fucked me so hard

- Method of successive Bisections. If:
  - f is continuous
  - We can find a st $f(a) > 0$
  - We can find b st $f(b) < 0$

  These values are determined by trial and error. Then by IVT, the root exists in between a and b! We can calculate the halfway point and call that x_{h1}. Then we can recursively perform this function to find the root.

- Newton's method is more sophisticated and is easier to use and is much faster. However, $f(x)$ must be differentiable and it doesn't always work.
  - Make a first guess for c, x_1.
  - Find equation for tangent line at $(x_1, f(x_1))$
  - Find x intercept of tangent line. Then let:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{293}$$

Warning: Note that sometimes it doesn't work! E.g. divergence $(x^{1/3})$

- General approach to find roots:
  - Try NM first
  - If x_n's converge, OK.
  - If not,try another.
  - If they still diverge, use MOSB.

Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$. For $f(x) = x^5$, $F(x) = \frac{1}{6}x^6 + C$.

17 Sigma Notation + Areas

- We begin by introducing sigma notation:

> **Definition**: If $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then:
> $$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \tag{294}$$

For example:

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^3 + 4^2 \tag{295}$$

There are a few theorems:

  - For a constant α:
$$\sum_{i=m}^{n} \alpha a_i = \alpha \sum_{i=m}^{n} a_i \tag{296}$$

  - It is also linear:
$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \tag{297}$$

  - $\sum_{i=1}^{n} \alpha = \alpha n$

  - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

  - $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

  - $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

- We can prove the last few properties via **induction**.

- We can also have limits inducing sums. For example, we can think of a sum as a function of n:

$$\sum_{i=1}^{n} a_i = f(n) \tag{298}$$

We can then think of the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \tag{299}$$

Since n is a parameter, it can appear in other parts of the sum as well.

Example 33: Evaluate $\lim_{n\to\infty}\left[\frac{5}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\right]$. We can evaluate this by separating: to get:

$$\lim_{n\to\infty}\left[\frac{5}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\right] = \lim_{n\to\infty}\left(\frac{5}{n^3}\sum_{i=1}^{n} i^2\right) \tag{300}$$

$$= \lim_{n\to\infty}\left(\frac{5}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \tag{301}$$

$$\frac{5}{3} \tag{302}$$

- Suppose we wish to solve the problem of determining the area under a curve. We can do this via a Riemann sum by approximating a curve as many sub-intervals.

Example 34: Suppose we wish to find the area under the curve of $x \in [0,1]$. Then if there are n rectangles, then the width of each rectangle is

$$\text{width} = \frac{1-0}{n} = \frac{1}{n} \tag{303}$$

The height for each of these is:

$$\frac{1}{n^2}, \frac{2^2}{n^2}, \ldots, \frac{n^2}{n^2} \tag{304}$$

so the total area is:

$$\text{Area} = \frac{1}{n}\frac{1}{n^2} + \frac{1}{n}\frac{2^2}{n^3} + \cdots + \frac{1}{n}\frac{n^2}{n^2} \tag{305}$$

$$= \frac{1}{n^3}(1 + 2^2 + 3^2 + \cdots + n^2) \tag{306}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \tag{307}$$

$$= \frac{n(n+1)(2n+1)}{6n^3} \tag{308}$$

$$= \frac{(n+1)(2n+1)}{6n^2} \tag{309}$$

$$\tag{310}$$

We can take the limit to find the area to be Area $= \frac{1}{3}$.

Definition: A partition is a finite subset of the closed interval $[a,b]$, which contains the points a and b. Denoted by P.

Definition: The norm of $P = \|P\|$ which is the length of the longest subinterval:

$$\|P\| = \max(\Delta x_1, \Delta x_2, \ldots, \Delta x_n) \tag{311}$$

- In general, the approximated total area under a curve becomes:

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_2 \tag{312}$$

And we let the largest subinterval go to zero:

$$A = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{313}$$

to get the total area.

Idea: In practice, our subintervals would be equal but this is not always the case in numerical methods.

Example 35: Let $y = \cos x$ and $0 \leq x \leq b \leq \frac{\pi}{2}$. To find the area, we can choose a regular partition:

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b}{n} = \|P\| \tag{314}$$

The right hand endpoints are:

$$x_i^* = x_i = \frac{ib}{n} \tag{315}$$

The area is thus:

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i \tag{316}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \cos\left(\frac{ib}{n}\right) \cdot \frac{b}{n} \tag{317}$$

We can apply the identity that:

$$\sum_{i=1}^{n} \cos(ix) = \frac{\sin(nx/2)\cos\left(\frac{x}{2}(n+1)\right)}{\sin(x/2)} \tag{318}$$

If we let $x = \frac{b}{n}$, then we get:

$$A = \lim_{n \to \infty} \frac{b}{n} \frac{\sin(b/2)\cos\left(\frac{(n+1)b}{2n}\right)}{\sin(b/2n)} \tag{319}$$

We can look at the limit involving the cosine term:

$$\lim_{n \to \infty} \cos\left(\frac{(n+1)b}{2n}\right) = \cos\left(\frac{b}{2}\right) \tag{320}$$

We can then make the substitution $t = \frac{b}{2n}$ to get:

$$\lim_{n \to \infty} \frac{b}{2n} \cdot \frac{2}{\sin(b/2n)} = \lim_{t \to 0+} 2 \cdot \frac{t}{\sin t} \tag{321}$$

$$= 2 \tag{322}$$

Similarly, we can find the limit of the denominator to get:

$$A = 2\sin(b/2)\cos(b/2) = \sin(b) \tag{323}$$

- Here is an example using a nonuniform partition.

Example 36: Evaluate $\int_0^2 x^{1/2} dx$. We can use a non-uniform partition:

$$x_i = i^2 \frac{2}{n^2} \tag{324}$$

such that:

$$\Delta x_i = x_i - x_{i-1} = \frac{2}{n^2}\left(i^2 - (i-1)^2\right) \tag{325}$$

For example for $n = 4$:

$$P = \left\{0, \frac{2}{16}, \frac{8}{16}, \frac{18}{16}, \frac{32}{16}\right\} \tag{326}$$

We can then approximate the area as:

$$A \approx \sum_{i=1}^{n} \Delta x_i f(x_i) \tag{327}$$

$$= \sum_{i=1}^{n} \frac{2}{n^2}\left(i^2 - (i-1)^2\right) \cdot \left(i^2 \frac{2}{n^2}\right)^{1/2} \tag{328}$$

$$= \sum_{i=1}^{n} \frac{2\sqrt{2}}{n^3}(i(i^3 - i^2 + 2i - 1)) \tag{329}$$

$$= \sum_{i=1}^{n} \frac{2\sqrt{2}}{n^3}(2i^2 - 1) \tag{330}$$

$$= \frac{2\sqrt{2}}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \tag{331}$$

where we have applied our properties. Taking the limit as $n \to \infty$ gives us:

$$A = \frac{2}{3}2^{3/2} \tag{332}$$

Note that we also have to check that each of our subintervals go to zero, or:

$$\lim_{n\to\infty} \|P\| = \Delta x_n = \frac{2}{n^2}\left(n^2 - (n-1)^2\right) = 0 \tag{333}$$

18 The Definite Integral

- We introduce the definite integral. We have already seen that:

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{334}$$

which arises when we find the area under a curve.

> **Definition**: If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition $x_0, x_1, x_2, \ldots, x_n$ where:
>
> $$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{335}$$
>
> Choose points x_i^* within each subinterval $[xi + 1, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral** of f from a to b is:
>
> $$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{336}$$
>
> if the limit exists. If the limit does exist, then f is called integrable on the interval $[a, b]$. The sign $\int$ is called the integral sign. $f(x)$ is known as the **integrand**, and a, b are the limits of integration. The output is a single number that does not depend on x.

- A Riemann Sum is:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \tag{337}$$

and we have used this to define an integral, but there are other definitions as well.

> **Warning**: The integral is not defined as the area under a curve, but it can be approximated as such.

> **Idea**: If we have:
>
> $$\int_a^b f(x)\,\mathrm{d}x = I \tag{338}$$
>
> then for ever $\epsilon > 0$, there exists a $\delta > 0$ such that:
>
> $$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \epsilon \tag{339}$$
>
> for all partitions P of $[a, b]$ with $\|P\| < \delta$ and all possible choices of x_i^* in $[x_{i-1}, x_i]$.

- If $b < a$, then:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{340}$$

and if $a = b$, then:

$$\int_a^b f(x)\,\mathrm{d}x = 0 \tag{341}$$

- How can we proved that the integral exists? We can use the theorem:

> **Theorem**: Continuous and/or piecewise continuous on $[a, b]$ guarantees integrability on $[a, b]$,

> **Definition**: A function is **piecewise continuous** if it only has a finite number of jump discontinuities.

- There are a few conventions to make Riemann sums more accessible:
  - We usually select regular partitions:

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{n} \tag{342}$$

  - And we select x_i^* to be the RH end point such that:

$$x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n} \tag{343}$$

Therefore, the integral can be written as:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \tag{344}$$

Example 37: Suppose we wish to evaluate $\int_0^3 (x^3 - 5x)\,\mathrm{d}x$. Then:

$$\begin{aligned} I &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 5\left(\frac{3i}{n}\right)\right] && (345)\\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{45}{n^2}\sum_{i=1}^{n} i\right] && (346)\\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{45}{n^2}\frac{n(n+1)}{2}\right] && (347)\\ &= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{45}{2}\left(1+\frac{1}{n}\right)\right] && (348)\\ &= \frac{81}{4} - \frac{45}{2} && (349)\\ &= -\frac{9}{4} && (350) \end{aligned}$$

Note that since right hand endpoints can be a problem, we can also define this by using a LH end point or even a midway endpoint.

- There are a few properties:
  - Constant:

$$\int_a^b c\,\mathrm{d}x = c(b-a) \tag{351}$$

  - Additivity:

$$\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x \tag{352}$$

  - Constant Multiple:

$$\int_a^b c(f)x\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \tag{353}$$

  - Changing Limits:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^z f(x)\,\mathrm{d}x + \int_z^b f(x)\,\mathrm{d}x \tag{354}$$

- There are also order properties of integrals. If $a < b$, then:
  - If $f(x) \geq 0$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,\mathrm{d}x \geq 0 \tag{355}$$
  - If $f(x) \geq g(x)$ for $a \leq x \leq b$, then:
$$\int_a^b f\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x \tag{356}$$
  - If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then:
$$m(b-a) \leq \int_a^b f\,\mathrm{d}x \leq M(b-a) \tag{357}$$
  - Absolute values:
$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x \tag{358}$$

19 The Fundamental Theorem of Calculus

- Suppose we have the new function:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{359}$$

where t is the dummy variable. For example, the area of a 45 degree triangle is:

$$\int_0^x t dt = \frac{1}{2}x^2 = F(x) \tag{360}$$

Notice that:

$$F'(x) = x = f(x) \tag{361}$$

- For $h > 0$, we can approxiamately write that:

$$F(x+h) - F(x) \simeq hf(x) \implies \frac{F(x+h) - F(x)}{h} \simeq f(x) \tag{362}$$

Theorem: Let f be continuous on $[a, b]$. The function F is defined on $[a, b]$ by:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{363}$$

is continuous on $[a, b]$, differentiable on (a, b) and has derivative

$$F'(x) = f(x) \tag{364}$$

for all $x \in (a, b)$.

Proof: For x and $x + h$ in (a, b),

$$F(x+h) - F(x) = \int_a^{x+h} f(x)\,\mathrm{d}t - \int_a^x f(x)\,\mathrm{d}t \tag{365}$$

$$= \int_a^x f(t)\,\mathrm{d}t + \int_x^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \tag{366}$$

$$= \int_x^{x+h} f(t)\,\mathrm{d}t \tag{367}$$

For $h \neq 0$, we have:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \tag{368}$$

We can separate it into cases. If $h > 0$, then we can write per the extreme value theorem the minimum value of f as $f(u) = m$ and the maximum value as $f(v) = M$ for $u, v \in [x, x+h]$ such that:

$$mh \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq Mh \tag{369}$$

or:

$$f(u)h \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v)h \tag{370}$$

which we can rewrite, after dividing through by h:

$$f(u) \leq \frac{F(x+h) - F(x)}{h} \leq f(v) \tag{371}$$

As $h \to 0$, we have $u \to x$ and $v \to x$. Therefore:

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \tag{372}$$

$$\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x) \tag{373}$$

which gives us:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \tag{374}$$

or:

$$\frac{d}{dx} \int_a^x f(t)\,\mathrm{d}t = f(x) \tag{375}$$

- Now we can come to the **fundamental theorem of calculus**

Theorem: Let f be continuous on $[a, b]$. If G is any antiderivative for f on $[a, b]$, then:

$$\int_a^b f(t)\,\mathrm{d}t = G(b) - G(a) \tag{376}$$

Proof: Given that $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f and given that G is an antiderivative, then:

$$F'(x) = G'(x) \implies F(x) = G(x) + C \tag{377}$$

We know that $F(a) = 0$, so $G(a) + C = 0$ or $C = -G(a)$, which gives:

$$\int_a^b f(t)\,\mathrm{d}t = F(b) = G(b) - G(a) \tag{378}$$

20 Indefinite Integrals and the net Change Theorem

- Recall that antiderivatives are not unique. If $F(x)$ is an antiderivative, then: $G(x) = F(x) + C$ is also an antiderivative.
- For example, the indefinite integral of:
$$\int x \, \mathrm{d}x = \frac{1}{2}x^2 + C \tag{379}$$
gives a family of curves. The textbook gives a list of common derivatives.
- We can alternatively write the fundamental theorem of calculus as:
$$\int_a^b F'(x) \, \mathrm{d}x = F(b) - F(a) \tag{380}$$
which can be interpreted as the net change of $F'(x)$. For example:
$$\Delta x = \int_a^b v(t) \, \mathrm{d}t \tag{381}$$
gives the displacement from $t = a$ to $t = b$.
- Recall that the chain rule for derivatives is given by:
$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{382}$$

Idea: We can extend this to integration. Suppose we have an integral in the form:
$$\int f(g(x))g'(x) \, \mathrm{d}x \tag{383}$$
If we let $u = g(x)$, then $du = g'(x)dx$. So we can simplify the integral to:
$$\int f(u) \, \mathrm{d}u = F(u) + C = F(g(x)) + C \tag{384}$$

Once we have the indefinite integral, we can use back substitution to find the definite integral. We can avoid this step using a change of variables.

Theorem:
$$\int_a^b f(g(x))g'(x) \, \mathrm{d}x = \int_{g(a)}^{g(b)} f(u) \, \mathrm{d}u \tag{385}$$

- We can also abuse symmetry.

21 Area Between Curves

- Suppose we wish to find the area between two curves $f(x)$ and $g(x)$. We can do this by partitioning the area into infinitesimally small rectangles:

$$\Delta A_i = [f(x_i^*) - g(x_i^*)]\Delta x_i \tag{386}$$

so that the area is given by:

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x_i \tag{387}$$

$$= \int_a^b [f(x) - g(x)]\,\mathrm{d}x \tag{388}$$

If we let $f(x) \geq g(x)$ when $x \in [a, b]$, then this gives the difference in their areas $A_1 - A_2$.

- If the condition $f(x) \geq g(x)$ is not satisfied, then we must break up the integral into multiple parts (if we interpret the area as having a positive area only). We can modify the area formula to be:

$$A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x \tag{389}$$

- Suppose we have a curve $x = f(y)$ and $x = g(y)$ instead. The area between $y = a$ and $y = b$ works in the same way:

$$A = \int_a^b |f(y) - g(y)|\,\mathrm{d}y \tag{390}$$

- If we instead have two inequalities, we are flexible to choose any method.

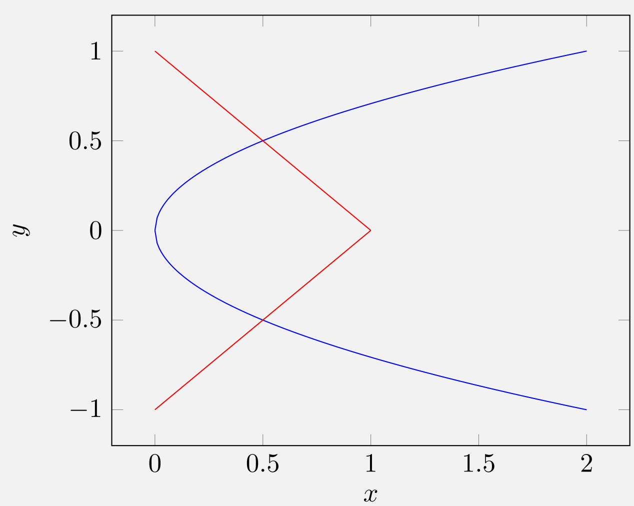
Example 38: What is the area contained within:

$$x - 2y^2 \geq 0 \tag{391}$$

$$1 - x - |y| \geq 0 \tag{392}$$

We can plot these two inequalities (sorry I don't know how to shade in regions in LaTeX):

Sample Problem



We can integrate with respect to y by noting that it is symmetric around the x axis. Then the two curves become:

$$x = 1 - y \tag{393}$$

$$x = 2y^2 \tag{394}$$

for $-\frac{1}{2} \leq y \leq 0$. Then the area is:

$$A = 2\int_0^{1/2} \left(1 - y - 2y^2\right) \mathrm{d}y \tag{395}$$

$$= 2\left(y - \frac{y^2}{2} - \frac{2y^3}{3}\right)\Big|_0^{1/2} \tag{396}$$

$$= 2\left(\frac{1}{2} - \frac{1}{8} - \frac{1}{12}\right) \tag{397}$$

$$= \frac{7}{12} \tag{398}$$

22 Volumes

- Suppose we have a cylinder whose axis is parallel to the x axis. Then we can break up the volume into thin sections:

$$V_i \simeq A_i \Delta x_i \tag{399}$$

so the volume is:

$$V = \int_a^b A(x)\,\mathrm{d}x \tag{400}$$

which is the general formula for the volume of any shape. If we can figure out $A(x)$ and the necessary bounds, we can find the volume for any change.

> **Example 39:** Suppose we wish to find the volume of a sphere. If we slice it up into small disks, then the area of each disk is:
>
> $$A = \pi\left(\sqrt{r^2 - x^2}\right)^2 \tag{401}$$
>
> then the volume is:
>
> $$V = \int_{-r}^{r} \pi(r^2 - x^2)\,\mathrm{d}x = \frac{4}{3}\pi r^3 \tag{402}$$

- For **solids of revolution**, imagine rotating the curve around the x-axis. Then the area of each disk is:

$$A(x) = \pi f(x)^2 \tag{403}$$

so using the disk method gives us the volume as:

$$V = \int_a^b \pi f(x)^2\,\mathrm{d}x \tag{404}$$

when the function $f(x)$ is rotated around the x axis.

- This even works for curves that cross the axis, since we end up squaring $f(x)$. For example, we can apply the same formula to $y = 2 - x^2$ from $x \in [0, 2]$.

- Additionally, we can use the disk method about the y-axis:

$$V = \int_c^d \pi g(y)^2\,\mathrm{d}y \tag{405}$$

- If we want the volume of revolution formed by rotating the region between two curves. The area between the two curves $f(x)$ and $g(x)$ is:

$$A = \pi\left(f(x)^2 - g(x)^2\right) \tag{406}$$

and we get:

$$V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right)\mathrm{d}x \tag{407}$$

known as the washer method. It works similarly for regions rotated about the y axis. Another way of looking at it is determining the volume of $f(x)$ rotated around the x axis and subtracting the volume of $g(x)$ rotated around the x axis from it.

- If we wish to find the volume bounded by two curves, then we need to find their intersection points first.

23 Volumes by Cylindrical Shells

- Sometimes, the washer method is difficult to apply. Suppose we wish to rotate a curve $f(x)$ from $x = a$ to $x = b$ around the y axis. Then we can look at a small rectangle with width Δx such that the volume of the cylindrical shell once rotated is:

$$V_i = f(x_i^*)\Delta x_i \cdot 2\pi x_i^* \tag{408}$$

so the volume is:

$$V = \int_a^b 2\pi x f(x)\,\mathrm{d}x \tag{409}$$

This is known as the **shell method about y-axis**

- Similarly we can apply this for a curve rotated about the x axis:

$$V = \int_a^b 2\pi y f(y)\,\mathrm{d}y \tag{410}$$

which is the **shell method about x-axis**.

Warning: Note that this is the opposite to the washer method. If we rotated across the x axis in this case, we integrate with respect to y.

Example 40: Find the volume bounded by $y^2 - x^2 = 1$ and $y = 2$ when they are rotated about the x axis.

It is essentially a hyperbola. Solving for x gives:

$$x = \pm\sqrt{y^2 - 1} \tag{411}$$

If we integrate with respect to y, then the bound is between $y = 1$ and $y = 2$. We can use symmetry and look at only positive values of x, then double it to get:

$$V = 2\int_1^2 (2\pi y)\sqrt{y^2 - 1}\,\mathrm{d}y \tag{412}$$

Using the u substitution of $u = y^2 - 1$, we get:

$$V = 4\sqrt{3}\pi \tag{413}$$

Example 41: Find the volume bounded by $y = x^2$ and $y = \sqrt{x}$ when rotated about the y axis.

We get:

$$V = \int_0^1 2\pi x(\sqrt{x} - x^2)\,\mathrm{d}x = \frac{3\pi}{10} \tag{414}$$

where I have omitted the intermediate steps, where we can just apply the power rules.

24 Average Value of a Function

- The average of a discrete set $\{a_1, a_2, \ldots, a_N\}$ is given by:

$$a_{\text{avg}} = \frac{1}{N}\sum_{i}^{N} a_i \tag{415}$$

- For a continuous distribution, we can extend this to:

$$f_{\text{avg}} = \frac{1}{N}\sum_{i}^{N} f(x_i^*) \tag{416}$$

Taking the limit as $N \to \infty$, we get:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \tag{417}$$

Theorem: **Mean Value Theorem for Integrals:** If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \tag{418}$$

Proof: Define $F(x) = \int_a^x f(t)\,\mathrm{d}t$. If we apply the mean value theorem to F, then:

$$F'(c) = \frac{F(b) - F(a)}{b-a} \tag{419}$$

for some $c \in [a,b]$. Now since:

$$F'(x) = f(x) \tag{420}$$

it becomes apparent that:

$$f(c) = \frac{\int_a^b f(t)\,\mathrm{d}t - \cancel{\int_a^a f(t)\,\mathrm{d}t}}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \tag{421}$$

Theorem: **Second Mean Value Theorem for Integrals:** Given f, g continuous on $[a,b]$ and g is non-negative, there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x \tag{422}$$

Proof: From the extreme value theorem (applies since $[a,b]$ is closed and bounded), there exists m, M such that

$$m \leq f(x) \leq M \tag{423}$$

and so

$$mg(x) \leq g(x)f(x) \leq Mg(x) \tag{424}$$

$$\int_a^b mg(x)\,\mathrm{d}x \int_a^b g(x)f(x)\,\mathrm{d}x \leq \int_a^b M(g)x\,\mathrm{d}x \tag{425}$$

with $x \in [a,b]$. This gives us

$$m\int_a^b g(x)\,\mathrm{d}x \leq \int_a^b g(x)f(x)\,\mathrm{d}x \leq M\int_a^b g(x)\,\mathrm{d}x \tag{426}$$

and dividing through, we can bound

$$m \leq \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \leq M \tag{427}$$

Since we also know that $m \leq f(x) \leq M$ when $x \in [a,b]$, we can apply the intermediate value theorem to show that there exists $x \in [a,b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \tag{428}$$

Warning: Note that $f(c) \neq f_{\text{avg}}$. It is not the average value of $f(x)$, but instead the *weighted* average of $f(x)$. This has applications in statistics, physics (i.e. center of mass), and other areas.

- Note that a direct corollary is that we can use the second mean value theorem to prove the first mean value theorem by setting $g(x) = 1$. Specifically, we get

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a) \tag{429}$$

- We can also introduce **inverse functions**.

Definition: A function $f(x)$ is said to be one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Alternatively, we can say that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

- We can use the **horizontal line test**. If any horizontal line crosses the function more than one time, then it is not one-to-one.

Definition: Let f be a 1-1 function with domain A and range B. Then its inverse function, f^{-1} has domain B and range A, and is defined by:

$$f^{-1}(x) = x \iff f(x) = y \tag{430}$$

Therefore:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \tag{431}$$

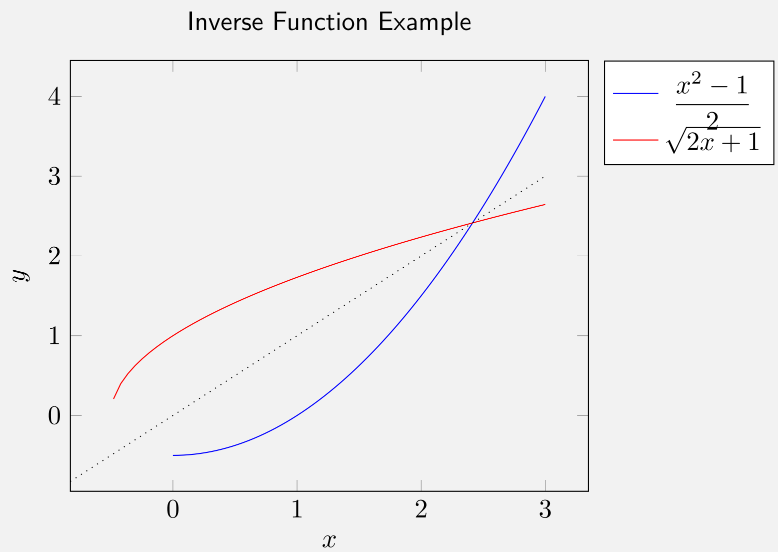
Warning: To prevent confusion, not that:

$$\frac{1}{f(x)} = [f(x)]^{-1} \neq f^{-1}(x) \tag{432}$$

- Geometrically, the inverse of a function represents a reflection of each point across the line $y = x$.

Example 42: If $g(x) = \sqrt{2x+1}$, it is implied that $x \geq -1/2$, so it is a one-to-one function. Therefore, the inverse function is:

$$g^{-1}(x) = \frac{x^2-1}{2} \tag{433}$$



Theorem: If f is either an increasing or decreasing function, then f is $1-1$, and hence, has an inverse.

Proof. Say $f(x)$ is decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$ and if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. ☐

Theorem: Let f be a 1-1 function defined on an interval I. If f is continuous, then f^{-1} is also continuous. (Proof provided in Appendix F)

- Let $g(x) = f^{-1}(x)$. Then:

$$f(g(x)) = x \tag{434}$$

$$\frac{d}{dx} f(g(x)) = 1 \tag{435}$$

$$f'(g(x))g'(x) = 1 \tag{436}$$

$$g'(x) = \frac{1}{f'(g(x))} \tag{437}$$

or:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{438}$$

which is equivalent to:

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} \tag{439}$$

Theorem: The inverse of composite functions is given by:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \tag{440}$$

Proof. Let $y = (f \circ g)^{-1}(x)$. Then:

$$x = (f \circ g)(y) = f(g(y)) \tag{441}$$

so we have:

$$g(y) = f^{-1}(x) \tag{442}$$

$$y = g^{-1}(f^{-1})(x) \tag{443}$$

$$= (g^{-1} \circ f^{-1})(x) \tag{444}$$

☐

25 The Natural Logarithm

- The logarithm is defined as:

> **Definition**: A logarithm function is a nonconstant differentiable function f defined for $x \in \{\mathbb{R}, (0, \infty)\}$ such that for all $a > 0$ and $b > 0$:
> $$f(a \cdot b) = f(a) + f(b) \tag{445}$$

- It has the following properties:
  - $f(1) = 0$
  - $f(1/x) = -f(x)$
  - $f(x/y) = f(x) - f(y)$
  - $f'(x) = \frac{1}{x} f'(1)$.

  The first three are trivial to prove, but the last one is not obvious:

> **Proof:** Define:
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{446}$$
> $$= \lim_{h \to 0} \frac{1}{h} f\left(\frac{x+h}{x}\right) \tag{447}$$
> $$= \lim_{h \to 0} \frac{f(1 + \frac{h}{x}) - f(1)}{x \cdot \frac{h}{x}} \tag{448}$$
> Let $k \equiv \frac{h}{x}$ such that the limit becomes:
> $$= \frac{1}{x} \lim_{k \to 0} \frac{f(1+k) - f(1)}{k} \tag{449}$$
> where the right hand side gives the derivative of $f(x)$ evaluated to $f'(1)$. If we choose a function $f'(1) = 1$, then it is known as the natural logarithm.

- This leads to the definition of the natural logarithm:

$$\ln(x) = \int_1^x \frac{\mathrm{d}t}{t} \tag{450}$$

  for $x > 0$. It has the following properties:
  - $\ln(x)$ is defined on $(0, \infty)$. For $x > 0$, it is strictly increasing.
  - $\ln(x)$ is continuous, since it is differentiable.
  - For $x > 1$, $\ln(x) > 0$.
  - For $0 < x < 1$, $\ln(x) < 0$.
  - $\ln(a \cdot b) = \ln(a) + \ln(b)$.
  - $\ln\left(x^{p/q}\right) = \frac{p}{q} \ln(x)$.
  - The range of $\ln(x)$ is from $(-\infty, \infty)$.

- There is a number e such that $\ln(e) = 1$.
- $\ln\left(e^{p/q}\right) = p/q$.
- For convention $\ln(x) = \log_e(x)$.
- $(\ln x)' = \dfrac{1}{x}$, increasing.
- $(\ln x)'' = -\dfrac{1}{x^2}$, concave down.

26 Feynman's Trick of Differentiation

- If we want to graph a function $\ln(f(x))$, the domain is given by x values such that $f(x) > 0$. As a result:

$$\int \ln(x)\,\mathrm{d}x \neq \frac{1}{x} \tag{451}$$

since x cannot be negative.

- Instead, we define:

$$\ln|x| = \frac{1}{x} \tag{452}$$

And similarly the following definite integral is defined as:

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C \tag{453}$$

Theorem: **Feynman's trick of Differentiation** (otherwise known as logarithmic differention): The following was popularized by Richard Feynman during the first of his Feynman Lectures. If we have a function:

$$g(x) = g_1(x)g_2(x)g_3(x)\cdots g_n(x) \tag{454}$$

Then taking the natural logarithm of both sides, applying the chain rule, and simplifying gives:

$$g'(x) = g(x)\left(\frac{g_1'}{g_1} + \frac{g_2'}{g_2} + \cdots + \frac{g_n'}{g_n}\right) \tag{455}$$

Example 43: Take the derivative of $g(x) = \dfrac{x^4(x-1)}{(x+2)(x^2+1)}$. While we can calculate this directly, we can also use the Feynman's trick to calculate the derivative: We get

$$g'(x) = \frac{x^4(x-1)}{(x+2)(x^2+1)}\left[\frac{4x^3}{x^4} + \frac{1}{x-1} - \frac{1}{x+2} - \frac{2x}{x^2+1}\right] \tag{456}$$

27 The Natural Exponential Function

- The exponential function can be written as:

$$\ln e^{p/q} = \frac{p}{q} \tag{457}$$

where p and q are integers. Thus, there must also be some number q such that:

$$\ln q = \pi \implies q = e^{\pi} \tag{458}$$

Definition: If z is irrational, then e^z is the unique number such that:

$$\ln(e^z) = z \tag{459}$$

Definition: The exponential function is written as:

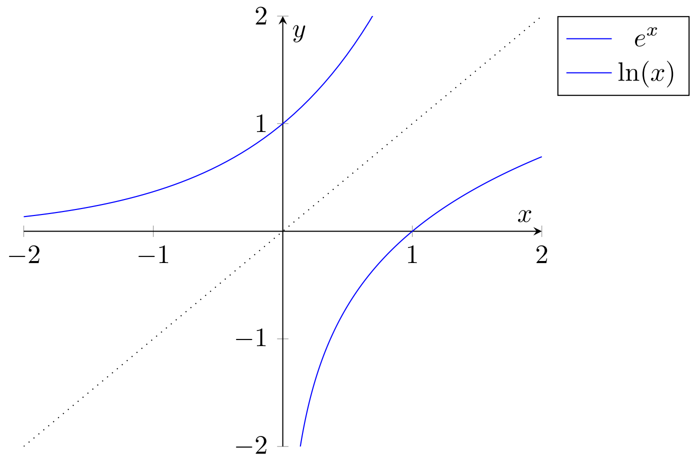
$$\exp(x) = e^x \tag{460}$$

for all real x.

- Here are the properties of e^x:

1. $\ln(e^x) = x$ where $x \in \mathbb{R}$
2. The graph looks like the following

Plot of e^x along with $\ln x$



3. $e^x > 0$

The exponential function is always greater than zero. Comes from the fact that it is the inverse of the natural logarithm function.

4. $e^0 = 1$
5. $\lim_{x \to -\infty} e^x = 0$
6. $e^{\ln x} = x$
7. $e^{a+b} = e^a e^b$ for all real a and b
8. $e^{-b} = \frac{1}{e^b}$
9. $\frac{d}{dx} e^x = e^x$

Proof:

$$\ln(e^x) = x \implies \frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\cdot\frac{d}{dx}(e^x) = 1$$

10. $\dfrac{d}{dx}e^{u(x)} = e^u\dfrac{du}{dx}$.

For example, $\dfrac{d}{dx}e^{kx} = ke^{kx}$ which is a relationship appears many times as this represents a relationship in which the rate of growth/decay is proportional to the actual amount.

11. $\displaystyle\int e^x \, \mathrm{d}x = e^x + C$

12. $\displaystyle\int e^{g(x)} g'(x) \, \mathrm{d}x = e^{g(x)} + C$

Example 44: Show $e^x > 1 + x$ for $x > 0$. We can show this via integration:

$$e^x = 1 + \int_0^x e^t \, \mathrm{d}t = 1 + e^x - e^0 = e^x \tag{461}$$

Note that $\dfrac{d}{dx}e^x > 0$ is always increasing. Therefore, we can claim that $e^x > 1$ for $x > 0$. Therefore:

$$1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x \mathrm{d}t = 1 + x \tag{462}$$

$$e^x > 1 + x \tag{463}$$

If we continue this, we will get:

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{464}$$

Try this yourself pls.

Example 45: Sketch the curve $f(x) = x^4 e^{-x}$. We go through our checklist:

- The domain of f is $\mathbb{R}$ and the range is $f \geq 0$.
- The derivative is

$$f' = 4x^3 e^{-x} + x^4(-1)e^{-x} = x^3 e^{-x}(4 - x) \tag{465}$$

  and is equal to $f' = 0$ when $x = 0, 4$. We have $f' < 0$ for $x > 4$ or $x < 0$ and $f' > 0$ for $0 < x < 4$.
- The local minimum is at $f(0) = 0$ and the local maximum is at $f(4) = 256e^{-4} \approx 4.2$.
- The second derivative is

$$f''(x) = 12x^2 e^{-x} - 4x^3 e^{-x} - 4x^3 e^{-x} + x^4 e^{-x} \tag{466}$$

$$= x^2 e^{-x}(x^2 - 8x + 12) \tag{467}$$

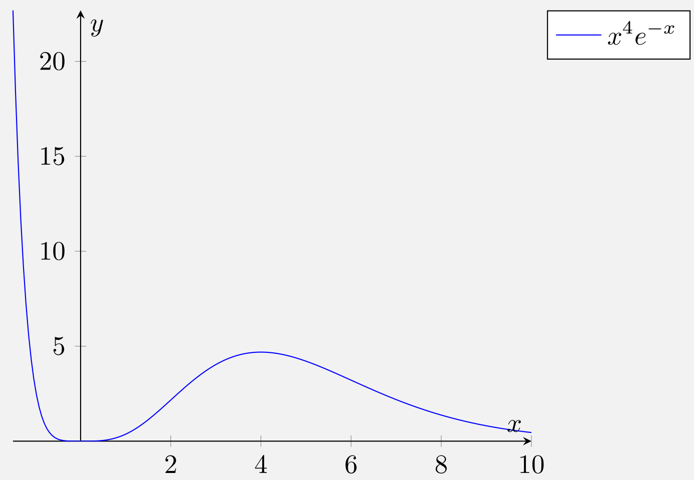
$$= x^2 e^{-x}(x - 6)(x - 2) \tag{468}$$

  We have:
  * $f'' = 0$ at $x = 0, 2, 6$.
  * $f'' > 0$ for $x > 6$ (concave up)
  * $f'' < 0$ for $2 < x < 6$ (concave down)
  * $f'' > 0$ for $x < 2$ (concave up)

  The points of inflection are at $x = 2$ and $x = 6$.

We can finally draw the picture:

Example Graph



Interestingly, this is part of a family of functions that have applications in medicine (among other fields).

Example 46: Suppose we wish to integrate

$$I = \int \frac{xe^{ax^2}}{e^{ax^2}+1}\,\mathrm{d}x \tag{469}$$

We motivate our solution by noting that the derivative of the denominator is contained in the numerator. Let $u = e^{ax^2}+1$, $\mathrm{d}u = 2axe^{ax^2}\,\mathrm{d}x$, such that

$$I = \frac{1}{2a}\int \frac{\mathrm{d}u}{u} = \frac{1}{2a}\ln\left(e^{ax^2}+1\right)+C \tag{470}$$

where we have removed the absolute value signs since the argument of the natural logarithm is always positive.

Example 47: Let us calculate $\displaystyle\int_0^{\sqrt{2\ln 3}} xe^{-x^2/2}\,\mathrm{d}x$. This type of integral will show up often in ECE286. Let $u = -\dfrac{1}{2}x^2$ and $\mathrm{d}u = -x\,\mathrm{d}x$. We can change the bounds to $u = 0$ (at $x = 0$) and $u = -\ln 3$ (at $x = \sqrt{2\ln 3}$). This gives

$$\int_0^{\sqrt{2\ln 3}} xe^{-x^2/2}\,\mathrm{d}x = -\int_0^{-\ln 3} e^u\,\mathrm{d}u \tag{471}$$

$$= -[e^u]_0^{-\ln 3} \tag{472}$$

$$= \frac{2}{3} \tag{473}$$

28 General Loagrithmic and Exponential Functions

- The general exponential function can be defined below:

Definition:

$$x^z = e^{z \ln x} \tag{474}$$

for $x > 0$

- As a result, we have the following properties:

$$x^{r+s} = x^r x^s \tag{475}$$
$$x^{r-s} = \frac{x^r}{x^s} \tag{476}$$
$$(x^r)^s = x^{rs} \tag{477}$$
$$\tag{478}$$

For $r, s \in \mathbb{R}$ and $x > 0$.

- We also have:

$$\frac{d}{dx} x^p = p x^{p-1} \tag{479}$$
$$\frac{d}{dx} x^r = r x^{r-1} \tag{480}$$

Example 48: Evaluate

$$I = \int \frac{x \, \mathrm{d}x}{(x^2+1)^{\sqrt{2}}} \tag{481}$$

Set $u = x^2 + 1$ such that $\mathrm{d}u = 2x \, \mathrm{d}x$ such that:

$$I = \frac{1}{2} \int \frac{\mathrm{d}u}{u^{\sqrt{2}}} \tag{482}$$
$$= \frac{1}{2} \left(\frac{1}{1-\sqrt{2}} \right) (x^2+1)^{1-\sqrt{2}} \tag{483}$$

Example 49: If $f(x) = x^x$, evaluate $f'(x)$.

We let:

$$x^x = e^{x \ln(x)} \tag{484}$$
$$(x^x)' = e^{x \ln(x)} \left(x \cdot \frac{1}{x} + \ln(x) \right) \tag{485}$$
$$= x^x (1 - \ln(x)) \tag{486}$$

- The derivative of $f(x) = p^x$ is given as:

$$\frac{d}{dx} p^u = p^u \ln(p) \frac{du}{dx} \tag{487}$$

Notice that the $\ln(p)$ term goes to one as $p \to e$.

- We can introduce logarithm functions to a base p:

$$f(x) = \frac{\ln(x)}{\ln(p)}, \, g(x) = p^x \tag{488}$$

such that:

$$f(g(x)) = \frac{\ln(p^x)}{\ln(p)} = x \tag{489}$$

so they are inverse functions.

Definition: The logarithm is defined as

$$\log_p(x) = \frac{\ln x}{\ln p} \tag{490}$$

Example 50:

$$\frac{d}{dx}\log_7(2x^3 - 2) = \frac{6x^2 - 1}{(2x^3 - x)\ln(7)} \tag{491}$$

- It is possible to estimate the value of e by bounding it. We have that:

$$\ln x = \int_1^x \frac{\mathrm{d}t}{t} \tag{492}$$

such that:

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} < \int_1^{1+1/n} 1\,\mathrm{d}t \tag{493}$$

Since $frac1t < \frac{1}{1}$ for $t > 0$. The upper bound then becomes:

$$1 + \frac{1}{n} - 1 = \frac{1}{n} \implies \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \tag{494}$$

We can similarly repeat this process:

$$1 + \frac{1}{n} < e^{1/n} \implies (1 + \frac{1}{n})^n < e \tag{495}$$

Note that if we take the limit as $n \to \infty$, *intuitively* we would expect the upper bound to become closer and closer to the true value. We shall explore this further, and we can write the lower bound as:

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} > \int_1^{1+1/n} \frac{\mathrm{d}t}{1 + 1/n} \tag{496}$$

since $\frac{1}{t} > \frac{1}{1 + 1/n}$. We can write this in logarithm form to get:

$$\ln\left(1 + \frac{1}{n}\right) > \left(\frac{1}{1 + 1/n}\right)\left(1 + \frac{1}{n} - 1\right) = \frac{1}{n+1} \implies \left(1 + \frac{1}{n}\right)^{n+1} > e \tag{497}$$

Putting it altogether, we have the following statement:

Idea: e can be estimated with its lower and upper bound with the following:

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \tag{498}$$

29 Inverse Trigonometric Functions

- We can define the inverse function of trigonometric functions by restricting their domain, such as from $-\pi/2$ to $\pi/2$ for $\sin(x)$.

Definition: The inverse function for $\sin(x)$ is given by :

$$\sin^{-1}(x) = \arcsin(x) \tag{499}$$

- It has the following properties such that:
  - $\sin^{-1}(\sin(x)) = x$ for $x \in [-\pi/2, \pi/2]$
  - $\sin\big(\sin^{-1}(x)\big) = x$ for $x \in [-1, 1]$

Warning: Note that these only work for the listed domains.

- Note that $\sin^{-1}(x) = -\sin^{-1}(x)$, so the inverse of $\sin(x)$ is also odd.
- We can also list the following properties:
  - $\cos\big(\sin^{-1}(x)\big) = \sqrt{1-x^2}$
  - $\tan\big(\sin^{-1}(x)\big) = \dfrac{x}{\sqrt{1-x^2}}$
  - $\sec\big(\sin^{-1}(x)\big) = \dfrac{1}{\sqrt{1-x^2}}$
  - $\cot\big(\sin^{-1}(x)\big) = \dfrac{\sqrt{1-x^2}}{x}$
  - $\csc\big(\sin^{-1}(x)\big) = \dfrac{1}{x}$
- The derivative of inverse sine can be calculated by considering the composite function:

$$\frac{d}{dx}\sin\big(\sin^{-1}(x)\big) = \frac{d}{dx} \tag{500}$$

$$\cos\big(\sin^{-1}(x)\big) \cdot \frac{d}{dx}(\sin^{-1}(x)) = 1 \tag{501}$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \tag{502}$$

for $x \in (-1, 1)$.

- A useful antiderivative:

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{503}$$

- We can similarly define the inverse tangent:

$$y = \tan^{-1}(x) \tag{504}$$

for $x \in (-\infty, \infty)$ and has a range of $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

- Some properties:
  - $\tan\big(\tan^{-1}(x)\big) = x$ for $x \in (-\infty, \infty)$
  - $\tan^{-1}(\tan(x)) = x$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

- Similarly, we can come up with the following composites by drawing a picture:
  - $\cot\left(\tan^{-1}(x)\right) = \dfrac{1}{x}$
  - $\sin\left(\tan^{-1}(x)\right) = \dfrac{x}{\sqrt{1+x^2}}$
  - $\cos\left(\tan^{-1}(x)\right) = \dfrac{1}{\sqrt{1+x^2}}$
  - $\sec\left(\tan^{-1}(x)\right) = \sqrt{1+x^2}$
  - $\csc\left(\tan^{-1}(x)\right) = \dfrac{\sqrt{1+x^2}}{x}$
- The derivative of the inverse tangent is:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \tag{505}$$

- Also know the following antiderivative:

$$\frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{506}$$

- Two useful results from the inverse secant function is:

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \tag{507}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C \tag{508}$$

30 Differential Equations

- A differential equation can be defined as:

> **Definition**: A differential equation is an equation which contains an unknown function with one or more of its derivatives.

- A ordinary differential equation refers to one independent variable.
- A partial differential equation refers to having two or more independent variables.
- The order of a differential equation refers to the highest derivative.

> **Definition**: The general solution refers to an n parameter family of solutions if they include all solutions to the differential equation.

> **Definition**: A particular solution refers to constants that are assigned particular values according to initial values, or boundary values.

- Not all differential equations have solutions, but separable ones can be written as:

$$\frac{dy}{dx} = F(x, y) = g(x)f(y) \tag{509}$$

For example:

$$\frac{dy}{dx} = \frac{1}{2}e^x y^2\,;y(0) = -1 \tag{510}$$

is a separable differential equation. Solving, we get:

$$\int \frac{2}{y^2}\,\mathrm{d}y = \int e^x\,\mathrm{d}x \tag{511}$$

such that the general solution is:

$$y = \frac{-2}{e^x + C} \tag{512}$$

and the particular solution is:

$$y = \frac{-2}{e^x + 1} \tag{513}$$

> **Idea**: In general, if the differential equation is in the form of:
>
> $$\frac{dy}{dx} = \frac{g(x)}{h(x)} \tag{514}$$
>
> then the solution can be written in the form of:
>
> $$\int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x \tag{515}$$

This can be verified by writing the function as:

$$h(y)\frac{dy}{dx} = g(x) \tag{516}$$

$$\int h(y)\frac{dy}{dx}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x \tag{517}$$

$$\frac{d}{dy}H(y) = h(y) \tag{518}$$

$$\frac{d}{dx}H(y) = h(y)\frac{dy}{dx} \tag{519}$$

$$\int h(y)\frac{dy}{dx} = \int \frac{d}{dx}H(y)\,\mathrm{d}x \tag{520}$$

$$H(y) = \int \frac{dH(y)}{dy}\,\mathrm{d}y \tag{521}$$

$$= \int h(y)\,\mathrm{d}y \tag{522}$$

$$\therefore \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x \tag{523}$$

Example 51: We can model the current in a resistor-inductor (RL) circuit if we know the energy dissipated by the resistor and inductor as:

$$V = IR\text{resistor}V = L\frac{dI}{dt} \qquad \text{inductor} \tag{524}$$

If the voltage source is a constant V, then the differential equation becomes:

$$V = L\frac{dI}{dt} + IR \tag{525}$$

and we can set the initial condition to be $I(0) = 0$. We can write this in the form:

$$\frac{dI}{dt} = \frac{V - RI}{L} \tag{526}$$

$$\int \frac{1}{V - IR}\,\mathrm{d}I = \int \frac{1}{L}\,\mathrm{d}t \tag{527}$$

$$-\frac{1}{R}\ln(V - IR) = \frac{t}{L} + C \tag{528}$$

$$V - IR = Ce^{-tR/L} \tag{529}$$

$$I = \frac{V}{R} - Ce^{-tR/L} \tag{530}$$

Note that the C value is not necessarily the same at each step. This is allowed as long as we only try to determine the value of C at the last step. For $R = 10\Sigma$, $L = 5$ H, and $V = 100$ V, we get:

$$I(t) = 10(1 - e^{-2t}) \tag{531}$$

as the particular equation.

- **Orthogonal trajectories** refer to curves that pass through a family of curves such that they remain perpendicular to each other such that:

$$f' = \frac{-1}{g'} \tag{532}$$

Example 52: Take the family of curves $y^2 = kx^3$. Using implicit differentiation, we get:

$$2yy' = 3kx^2 \implies y' = \frac{3kx^2}{2y} \tag{533}$$

Since we also have $k = \frac{y^2}{x^3}$, we get:

$$\frac{dy}{dx} = \frac{3}{2}\frac{y}{x} \tag{534}$$

The curve that is perpendicular to this original curve is thus described by the differential equation:

$$\frac{dy}{dx} = -\frac{2x}{3y} \tag{535}$$

Solving it gives:

$$3y^2 + 2x^2 = 2C \tag{536}$$

which represents a family of ellipses.

31 Exponential Growth and Decay

- There are many instances in physics and nature where the growth of a function is related to the function at that point, such as:

$$\frac{df}{dt} = kf(t) \implies k = \frac{1}{f}\frac{df}{dt} = \frac{d}{dt}(\ln f) \tag{537}$$

Separating, we get:

$$\ln(f) + kt + C \implies f = Ce^{kt} \tag{538}$$

where C is based off initial conditions.

- The **doubling time** refers to the time for a function to double:

$$2P_0 = P_0e^{kt_2} \implies t_2 = \frac{\ln 2}{k} \tag{539}$$

- In many areas (such as radioactive decay), the half life gives the time necessary for the function to half. This occurs in functions where the DE looks like:

$$\frac{df}{dt} = -kN \tag{540}$$

where $k > 0$. Similarly, the half life is given by:

$$t_{1/2} = \frac{\ln 2}{k} \tag{541}$$

- For compound interest, the annual interest is given by:

$$V(t) = V_0(1+i)^t \tag{542}$$

If we compound the interest more and more often, we get:

$$V(t) = V_0\left(1+\frac{i}{n}\right)^{nt} \tag{543}$$

Taking the limit as $n \to \infty$, we get:

$$\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{nt} = V_0\lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^m\right)^{it} \tag{544}$$

$$= V_0e^{it} \tag{545}$$

where we made the substitution $m = n/i$.

- The **logistic model** is a more realistic model for population growth:

$$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right) \tag{546}$$

where M is the carry capacity or max population:

$$\int\frac{\mathrm{d}P}{P(1-P/M)} = k\int\mathrm{d}t \tag{547}$$

$$\int\left(\frac{1}{P}+\frac{1}{M-P}\right)\mathrm{d}P = k\int\mathrm{d}t \tag{548}$$

$$\ln|P| - \ln|M-P| = kt + C \tag{549}$$

$$\ln\left|\frac{P}{M-P}\right| = kt + C \tag{550}$$

$$\frac{P}{M-P} = \pm e^{kt+C} \tag{551}$$

$$P(t) = \frac{M}{1+Ae^{-kt}} \tag{552}$$

where $A \equiv \dfrac{M-P_0}{P_0}$ and $P_0 = P(t=0)$.

32 Linear Equations

- We introduce linear differential equations

Definition: A **linear first order differential equation** is in the form of:

$$y' + p(x)y = q(x) \tag{553}$$

where $p(x)$, and $q(x)$ are continuous on the interval I.

Example 53: Take the differential equation:

$$xy' + y = x^2 \tag{554}$$

Notice that the left hand side is the result of the product rule, so:

$$(xy)' = xy' + y \tag{555}$$
$$(xy)' = x^2 \tag{556}$$
$$\int \mathrm{d}xy = \int x^2 \,\mathrm{d}x \tag{557}$$
$$xy = \frac{x^3}{3} + C \tag{558}$$
$$y = \frac{x^2}{3} + \frac{C}{x} \tag{559}$$

- In general, we have to work a bit harder. We can define:

$$H(x) = \int p(x) \,\mathrm{d}x \tag{560}$$

Note that we can set the integration constant to zero since this is not the solution, but just a helpful quantity. We can then exponentiate and take the derivative:

$$\frac{d}{dx}e^{H(x)} = e^{H(x)}\frac{d}{dx}H(x) \tag{561}$$
$$= e^{H(x)}p(x) \tag{562}$$
$$\frac{d}{dx}(ye^{H(x)}) = y'e^{H(x)} + ye^{H(x)}p(x) \tag{563}$$
$$= e^{H(x)}(y' + p(x)y) \tag{564}$$

Note that the right factor is the LHS of the general differential equation. The other factor $e^{H(x)}$ is known as the **integrating factor.** As a result:

$$\frac{d}{dx}(ye^{H(x)}) = e^{H(x)}q(x) \tag{565}$$
$$ye^{H(x)} = \int e^{H(x)}q(x) \,\mathrm{d}x + C \tag{566}$$
$$y = e^{-H(x)}\left[\int e^{H(x)}q(x) \,\mathrm{d}x + C\right] \tag{567}$$

Idea: In general, the solution to

$$y' + p(x)y = q(x) \tag{568}$$

is

$$y = e^{-H(x)}\left[\int e^{H(x)}q(x) \,\mathrm{d}x + C\right] \tag{569}$$

Example 54: Suppose $y' + 2y = 4$. Then we can let $p(x) = 2$ and $q(x) = 4$. Therefore,

$$H(x) = \int 2\,\mathrm{d}x = 2x \tag{570}$$

such that:

$$e^{2x} \tag{571}$$

is the integrating factor. Finally, we have:

$$\int e^{H(x)} q(x)\,\mathrm{d}x = \int 4e^{2x}\,\mathrm{d}x = 2e^{2x} \tag{572}$$

so that the general solution is:

$$y = e^{-2x}\left(2e^{2x} + C\right) = 2 + Ce^{-2x} \tag{573}$$

Idea: Note that the solution has two terms, each representing a different solution. $y = 2$ represents a solution to:

$$y' + 2y = 4 \tag{574}$$

while Ce^{-2x} represents a solution to:

$$y' + 2y = 0 \tag{575}$$

This is a surprise tool that will come in handy later.

Example 55: Suppose:

$$y' - 4y = 3e^{x}y^{1/2} \tag{576}$$

This is not in the usual form but we can turn it into such with the substitution:

$$u = \sqrt{y} \implies u' = \frac{1}{2}y^{-1/2}y' \tag{577}$$

which gives:

$$u' - 2u = \frac{3}{2}e^{x} \tag{578}$$

- In general, an equation in the form of:

$$y' + p(x)y = q(x)y^{r} \tag{579}$$

with $r \neq 0, 1$ can be substituted using $u = y^{1-r} \implies u' + (1-r)p(x)u = (1-r)q(x)$. Equations in this form is known as **Bernoulli Equations**

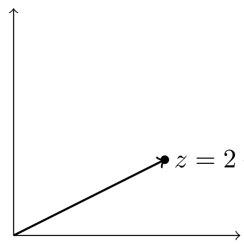
33 Complex Numbers

- We can introduce complex numbers to assign values to the solutions of algebraic equations such as:

$$x^2 = -1 \tag{580}$$

Definition: A complex number is defined as $z = a + ib$ where $a, b \in \mathbb{R}$ and $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

- We can represent complex numbers on a plane:



- It is often helpful to write out a complex number using polar coordinates. The **modulus** of the number is:

$$|z| = |a + ib| = \sqrt{a^2 + b^2} \tag{581}$$

and the argument is the angle it makes with the real axis:

$$\arg(z) = \theta + 2k\pi \tag{582}$$

where k is an integer. This means that:

$$|z|\cos(\theta) = a$$
$$|z|\sin(\theta) = b$$

Idea: The **polar representation** can be written as:

$$z = r\left(\sin\cos\theta + i\sin\theta\right) \tag{583}$$

where $r = |z|$.

- The **complex conjugate** for a complex number $z = a + ib$ is:

$$\bar{z} = a - ib \tag{584}$$

- Let $z_1 = a + ib$ and $z_1 = c + id$. Then complex addition/subtraction has the following properties:
  - $z_1 + z_2 = (a + c) + i(b + d)$
  - $z_1 + z_2 = z_2 + z_1$ (commutative)
  - $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (associative)
  - $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)
  - $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

- Complex multiplication can be defined as:

$$(a + ib)(c + id) = (ab - bd) + i(ad + bc) \tag{585}$$

It has the following properties:

  - $z_1 \cdot z_2 = z_2 \cdot z_1$ (commutative)

- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (associative)
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributive)
- $\overline{z_1 z_2} = \bar{z_1} \cdot \bar{z_2}$

Idea: When multiplying two complex numbers in their polar form, we get:

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_\phi) + i \sin(\theta + \phi)\right) \tag{586}$$

Note that:

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2) \tag{587}$$

and the modulus is:

$$|z_2 z_2| = |z_1||z_2| \tag{588}$$

What this means is that the magnitudes get multiplied like scalars and z_1 is rotated by the argument of z_2.

- One direct consequence of this idea is that multiplying by i is equivalent to rotating counterclockwise a complex number by 90 degrees. Note that this is an important concept that will appear when dealing with phasors in the circuit course.

Theorem: **De Moivre's Theorem:** Let $z = \cos\theta + i \sin\theta$. We have $|z| = 1$ and $\arg(z) = \theta$. Then:

$$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta) \tag{589}$$

Definition: We can define division by multiplying the denominator by its conjugate:

$$\frac{1}{z} = \frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} + \frac{\bar{z}}{|z|^2} \tag{590}$$

Therefore:

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \tag{591}$$

and:

$$\arg\left(\frac{1}{z}\right) = -\arg(z) \tag{592}$$

- The most important tool in working with complex numbers is the complex exponential:

$$z = e^{ix} \tag{593}$$

We cannot define this by making the following observation. Note that the derivative of $f(x) = eix$ is:

$$f'(x) = ie^{ix} = if(x) \tag{594}$$

and $f(0) = 1$. If we define $g(x) = \cos(x) + i \sin(x)$, then:

$$g'(x) = -\sin(x) + i \cos(x) = ig(x) \tag{595}$$

and $g(0) = 1$ also. Therefore, it seems convincing that $f(x) = g(x)$ or:

$$e^{ix} = \cos(x) + i \sin(x) \tag{596}$$

This is not a complete proof however, but will be rigorously proved next semester by using a Taylor series.

34 Second Order Differential Equations

- A typical **second order linear equation** takes the form of:

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = g(x) \tag{597}$$

However we will only look at cases with **constant coefficients**, that is equations in the form of:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = g(x) \tag{598}$$

- **Homogeneous 2nd order linear differential equations** with constant coefficients are in the form of:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{599}$$

Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of a homogeneous second order linear differential equation and c_1, c_2 are any constants, then the linear combination:

$$y(x) = C_1y_1(x) + C_2y_2(x) \tag{600}$$

is also a solution.

Proof: We have:

$$(c_1y_1 + c_2y_2)'' + a(c_1y_1 + c_2y_2)' + b(c_1y_1 + c_2y_2) = 0 \tag{601}$$
$$c_1(y_1'' + ay_1' + by_1) + c_2(y_2'' + ay_2' + by_2) = 0 \tag{602}$$
$$c_1(0) + c_2(0) = 0 \tag{603}$$

Theorem: If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a homogeneous second order linear differential equation, then:

$$y(x) = C_1y_1(x) + C_2y_2(x) \tag{604}$$

is the general solution. Two solutions are linearly independent iff:

$$y_2(x) \neq Cy_1(x) \tag{605}$$

- To solve this homogeneous second order differential equation:

$$y'' + ay' + by = 0 \tag{606}$$

We might *guess* a solution in the form of $y = e^{rx}$, to get:

$$(e^{rx})'' + a(e^{rx})' + b(e^{rx}) = 0 \implies (r^2 + ar + b)e^{rx} = 0 \tag{607}$$

Here, the **characteristic equation** (also referred to as the auxillary equation) is:

$$r^2 + ar + b = 0 \tag{608}$$

- There are three cases:
  - Case One: $a^2 - 4b > 0$: Then r_1, r_2 are real and distinct so the general solution is:

$$y = C_1e^{r_1x} + C_2e^{r_2x} \tag{609}$$

  - Case Two: $a^2 - 4b = 0$: Then $r_1 = r_2 = -\frac{a}{2} = r$. Then the solution is:

$$y = e^{rx} + xe^{rx} \tag{610}$$

- Case Three: $a^2 - 4b < 0$: Then $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ where $\alpha = -\frac{\alpha}{2}$ and $\beta = \frac{1}{2}\sqrt{4b - a^2}$. Using the complex identity, we can rewrite this as:k

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \tag{611}$$
$$= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \tag{612}$$
$$= e^{\alpha x}((C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x) \tag{613}$$
$$= e^{\alpha x}(A\cos\beta x + B\sin\beta x) \tag{614}$$

where the coefficients could either be real or complex. Typically, we only look at the real part when dealing with boundary conditions that only look at the real part.

- Initial value problems need two conditions, $y(x_0) = y_0$ and $y'(x_0) = y_1$. They will always have a solution (for our purposes).
- Boundary value problems have two differential conditions, such as: $y(x_0) = y_0$ and $y(x_1) = y_1$ OR $y'(x_0) = y_0$ and $y'(x_1) = y_1$. They will not always have a solution.

Example 56: Suppose $y'' + 4y' + 5y = 0$ and the following boundary conditions: $y(0) = 1$ and $y(\pi/2) = 0$. The characteristic equation is:

$$r^2 + re + 5 \implies r = -2 \pm i \tag{615}$$

so the general solution is:

$$y = e^{-2x}(A\cos x + B\sin x) \tag{616}$$

Using the initial conditions, we get $A = 1$ and $B = 0$. Therefore:

$$y = e^{-2x}\cos(x) \tag{617}$$

However, note that if the second boundary condition was $y(\pi) = 0$, which would have resulted in $A = 0$, but is a direct contradiction of $y(0) = 1$.

35 Non homogenous Linear Equations

- We now have the tools to solve the nonhomogeneous linear equation:

$$y'' + ay' + by' = \phi(x) \tag{618}$$

We can define the **complementary equation** to be:

$$y'' + ay' + by = 0 \tag{619}$$

Theorem: The general solution of a nonhomogeneous second order linear differential equation gwith constant coefficients is given by:

$$y(x) = y_p(x) + y_c(x) \tag{620}$$

where $y_p(x)$ is a particular solution of the complete differential equation and $y_c(x)$ is the general solution of the complementary homogeneous equation.

Proof. Given $y_{p_1}(x)$ and $y_{p_2}(x)$, let $z = y_{p_1} - y_{p_2}$ such that:

$$\implies y_{p_2} = y_{p_1} - z \tag{621}$$

$$y'_{p_2} = y'_{p_1} - z' \tag{622}$$

$$y''_{p_2} = y''_{p_1} - z'' \tag{623}$$

□

so that:

$$y'_{p_2} + ay'_{p_2} + by_{p_2} = \phi(x) \tag{624}$$

$$(y''_{p_1} - z'') + a(y'_{p_1} - z') + b(y_{p_1} - z) = \phi(x) \tag{625}$$

$$[y''_{p_1} + ay'_{p_1} + by_{p_1}] - [z'' + az' + bz] = \phi(x) \tag{626}$$

$$\phi(x) - [z'' + az' + bz] = \phi(x) \tag{627}$$

$$z'' + az' + bz = 0 \tag{628}$$

Therefore, z is a solution of the complementary homogeneous equation.

- The idea behind the **method of undetermined coefficients** is to assume that the undetermined function has the same form as $\phi(x)$. For example, take:

$$y'' - 6y' + 8y = x^2 + 2x \tag{629}$$

Solving the auxillary equation $r^2 - yr + 8 = 0$ gives two roots: $y_1 = C_1e^{2x} + C_2e^{4x}$. For the particular solution, we also assume that y has the same form as $\phi(x)$ which is a second order polynomial:

$$y_p = Ax^2 + Bx + C \tag{630}$$

Therefore, we get:

$$2A - 6(2Ax + B) + 8(Ax^2 + Bx + C) = x^2 + 2x \tag{631}$$

which after solving gives $A = 1/8$, $B = 7/16$, and $C = 19/64$. Therefore, we would get:

$$y = C_1e^{2x} + C_2e^{4x} + \frac{1}{8}x^2 + \frac{7}{17}x + \frac{19}{64} \tag{632}$$

- We can extend this to equations in the form of:

$$y'' + ay' + by = \phi_1(x) + \phi_2(x) \tag{633}$$

we can apply the superposition principle to determine the particular solution to be the particular solution to $\phi_1(x)$ added to the particular solution to $\phi_2(x)$.

- A neat trick occurs when the complementary solution resembles the form of $\phi(x)$, such as

$$y'' + y = \sin(x) \tag{634}$$

The solution of the complementary equation is $y_c = C_1\cos(x) + C_2\sin(x)$, so instead of trying $y_p = A\sin(x)$, we should try $y_p = Ax\cos(x) + Bx\sin(x)$ instead.

Warning: Note that we need to multiply it by a factor of x to prevent redundancy.

Example 57: Solve $y'' - y' - 6y = e^{-2x}$. We first get the auxillary equation:

$$r^2 - r - 6 = 0 \implies r = -2, 3 \tag{635}$$

so the complementary equation is:

$$y_c = C_1e^{-2x} + C_2e^{3x} \tag{636}$$

and we try the particular equation in the form of:

$$y_p = Axe^{-2x} \tag{637}$$

to get:

$$y_p = Axe^{-2x} \tag{638}$$
$$y_p' = Ae^{-2x} - 2Axe^{-2x} \tag{639}$$
$$y_p'' = (4Ax - 4A)e^{-2x} \tag{640}$$

We want:

$$(Ax - 4A)e^{-2x} - (A - 2Ax)e^{-2x} - 6Axe^{-2x} = e^{-2x} \tag{641}$$

which we can solve by matching coefficients. To ensure that the coefficients for the xe^{-2x} terms sum up to zero, we have:

$$4A + 2A - 6A = 0 \implies 0 = 0 \tag{642}$$

which means this is always satisfied. Matching the coefficients for the e^{-2x} terms, we get:

$$-4A - A = 1 \implies A = -\frac{1}{5} \tag{643}$$

Therefore, the solution is:

$$y = C_1e^{-2x} + C_2e^{3x} - \frac{1}{5}xe^{-2x} \tag{644}$$

- We can also use the methods of **variation of parameters** since guessing may not always be the most reliable. To begin, we look at the general solution of the homogeneous second order equation:

$$y'' + ay' + by = 0 \tag{645}$$

is:

$$y_c = c_1y_1(x) + c_2y_2(x) \tag{646}$$

Instead of having c_1 and c_2 are constants, we can let them be functions $u_1(x)$ and $u_2(x)$ when solving the nonhomogeneous equation:

$$y'' + ay' + by = \phi(x) \implies y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{647}$$

Note that this means:

$$y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2) \tag{648}$$

To simplify things, we can arbitrarily choose:

$$u_1'y_1 + u_2'y_2 = 0 \tag{649}$$

This gives:

$$y_p' = u_1y_1' + u_2y_2' \tag{650}$$

The second derivative is now:

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' \tag{651}$$

The advantage of this is that there are no second derivatives of u_1 or u_2. Substituting into the original equation gives:

$$(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2') + a(u_1 y_1' + u_2 y_2') + b(u_1 y_1 + u_2 y_2) = \phi(x) \tag{652}$$

We can rearrange them to be in the form of:

$$(y_1'' + a y_1' + b y_1)u_1 + (y_2'' + a y_2' + b y_2)u_2 + (u_1' y_1' + u_2' y_2') = \phi(x) \tag{653}$$

The first two terms evaluate to zero since y_1 and y_2 are solutions to the complementary equation, so we are left with:

$$u_1' y_1' + u_2' y_2' = \phi(x) \tag{654}$$
$$u_1' y_1 + u_2' y_2 = 0 \tag{655}$$

We have two equations and two unknowns and we just need to solve for $u_1'(x)$ and $u_2'(x)$.

Example 58: Take the differential equation $y'' + y' - 2y = e^x$. The auxillary equation gives:

$$r^2 + r - 2 = 0 \implies r = 1, -2 \tag{656}$$

This means that $y_1 = e^x$ and $y_2 = e^{-2x}$. We then get the system of equations:

$$u_1'(e^x) + u_2'(-2e^{-2x}) = e^x \tag{657}$$
$$u_1'(e^x) + u_2'(e^{-2x}) = 0 \tag{658}$$

Solving for u_1' gives $u_1' = \dfrac{1}{3}$ such that $u_1 = \dfrac{x}{3}$. Similarly, solving for u_2' gives:

$$u_2' = -\frac{1}{3}e^{3x} \implies u_2 = -\frac{1}{8}e^x \tag{659}$$

So the particular solution is:

$$\frac{1}{3}xe^x - \frac{1}{9}e^x \tag{660}$$

The second term can be included as one of the terms in the complementary solution, so the general solution is:

$$y = C_1 e^x + C_2 e^{-2x} + \frac{1}{3}xe^x \tag{661}$$

Example 59: Take the differential equation:

$$y'' + y = 3\sin x \sin 2x \tag{662}$$

The auxillary equation leads to:

$$r^2 + 1 = 0 \implies r = \pm i \tag{663}$$

which gives the complementary solution as:

$$y_c = A\cos x + B\sin x \tag{664}$$

such that we have:

$$y_p = u_1(x)\cos x + u_2(x)\sin(x) \tag{665}$$

And solving the system:

$$u_1'\cos x + u_2'\sin x = 0 \tag{666}$$
$$-u_1'\sin x + u_2'\cos x = 3\sin x\sin 2x \tag{667}$$

gives:

$$u_1' = -3\sin^2 x\sin(2x) \tag{668}$$

and:

$$u_2' = 3\cos x \sin x \sin 2x \tag{669}$$

Integrating these gives:

$$u_1 = -\frac{3}{2}\sin^4 x \tag{670}$$

$$u_2 = \frac{3}{4}\left(x - \frac{1}{4}\sin(4x)\right) \tag{671}$$

The particular solution then becomes:

$$y_p = -\frac{3}{2}\cos x \sin^4 x + \frac{3}{16}(4x - \sin 4x)\sin x \tag{672}$$

and the general solution is:

$$y = A\cos(x) + B\sin(x) - \frac{3}{2}\cos x \sin^4 x + \frac{3}{16}(4x - \sin 4x)\sin x \tag{673}$$

Theorem: In general, if we have a differential equation in the form of:

$$y'' + ay' + by = \phi(x) \tag{674}$$

then the complementary solution is given as:

$$y_c = Ay_1(x) + By_2(x) \tag{675}$$

where $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$, and r_1, r_2 are the solutions to the quadratic:

$$r^2 + ar + b = 0 \tag{676}$$

except for the case of a double root, which you should see lecture 34 for. The particular solution is given by:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{677}$$

where $u_1'(x)$ and $u_2'(x)$ are given by:

$$u_1'(x) = \frac{-y_2\phi(x)}{y_1y_2' - y_2y_1'} \tag{678}$$

$$u_2'(x) = \frac{y_1\phi(x)}{y_1y_2' - y_2y_1'} \tag{679}$$

Integrating and letting the constant of integration to be zero, we can solve for $u_1(x)$ and $u_2(x)$. The general solution is then:

$$y = y_c(x) + y_p(x) \tag{680}$$