ESC194: Midterm 1 Review

QiLin Xue

Fall 2020

Contents

Index

1 Delta-Epsilon Proofs

1.1 Brief Overview

The formal definition of the limit $\lim_{x \to c} f(x) = L$:

> **Definition**: If for any $\epsilon > 0$, a $\delta > 0$ can be found such that for all $0 < |x - c| < \delta$, it can be proved that $|f(x) - L| < \epsilon$, then $\lim_{x \to c} f(x) = L$.

The *general steps* are as follows:

- Write: "For any $\epsilon > 0$, we want to pick a $\delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$"
- Start with $|f(x) - L| < \epsilon$ to start getting it under δ control (e.g. by expressing the LHS in terms of δ)
- Pick an arbitrary value of $\delta = a$ (if in doubt, choose $a = 1$) and modify $0 < |x - c| < a$ to write x in terms of a. Substitute this back into $|f(x) - L| < \epsilon$ to fully express the LHS in terms of δ.
- Solve for δ in terms of ϵ and pick $\delta = \min\{a, f(\epsilon)\}$.

A few *tips/tricks:*

- Apply the **Triangle Inequality:** $|a + b| \leq |a| + |b|$.
- Apply the identity: $|ab| = |a||b|$.
- Apply the inequality: $\frac{1}{x} > \frac{1}{x + a}$ for $x > 0$ given $a > 0$.
- Remember that $0 < |x - c| < \delta \implies c - \delta < x < c + \delta$.

Example 1: (2019 Midterm, Modified) Prove $\lim_{x \to 2} \frac{3x + 1}{(x + 1)^2} = 1$.

For any $\epsilon > 0$, we want to pick a $\delta > 0$ such that $0 < |x - 2| < \delta \implies \left| \frac{3x + 1}{(x + 1)^2} - 1 \right| < \epsilon$. We can start with:

$$\left| \frac{3x+1}{(x+1)^2} - 1 \right| < \epsilon \implies \left| \frac{3x + 1 - (x^2 + 2x + 1)}{(x+1)^2} \right| \tag{1}$$

$$\implies \left| \frac{x - x^2}{(x+1)^2} \right| < \epsilon \tag{2}$$

$$\implies \left| \frac{x(1 - x)}{(x+1)^2} \right| < \epsilon \tag{3}$$

$$\implies \left| \frac{x}{(x+1)^2} \right| |x - 1| < \epsilon \tag{4}$$

$$\implies \left| \frac{x}{(x+1)^2} \right| |(x - 1 - 1) + (1)| < \epsilon \tag{5}$$

$$\implies \left| \frac{x}{(x+1)^2} \right| (|x - 2| + |1|) < \epsilon \tag{6}$$

$$\implies \left| \frac{x}{(x+1)^2} \right| (\delta + 1) < \epsilon \tag{7}$$

$$\tag{8}$$

We can set $\delta = 1$. If this is the case then:

$$0 < |x - 2| < 1 \implies 1 < x < 3 \iff 2 < x + 1 < 4 \tag{9}$$

We can bound the denominator $|(x+1)^2|$ by its lower bound $2^2 = 4$ and the numerator $|x|$ by its upper bound of 3, which we can substitute back in to get:

$$\left|\frac{x}{(x+1)^2}\right|(\delta+1) < \frac{3}{4}(\delta+1) \le \epsilon \implies \delta \le \frac{4}{3}\epsilon - 1 \tag{10}$$

Thus, we can pick:

$$\delta = \min\{1, \frac{4}{3}\epsilon - 1\} \tag{11}$$

and we are done. Note that we could also have applied the identity $\frac{1}{x} > \frac{1}{x+a}$ to bound the denominator by 1^2 instead.

1.2 Special Limits

For right handed limit, we have:

Definition: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that $c < x < c + \delta \implies |f(x) - L| < \epsilon$, then $\lim_{x \to c^+} = L$.

For left handed limits:

Definition: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that $c - \delta < x < c \implies |f(x) - L| < \epsilon$, then $\lim_{x \to c^-} = L$.

For infinite limits:

Definition: If for every $M > 0$, a $\delta > 0$ can be found such that $0 < |x - c| < \delta \implies f(x) > M$, then $\lim_{x \to c} = \infty$.

Here's an example using both:

Example 2: (2019 Quiz 2H, Modified) Prove the infinite limit $\lim_{x \to 2^+} \frac{x^{3/2}}{(x-2)^2} = \infty$.

For any $M > 0$, we want to pick a $\delta > 0$ such that $2 < x < 2 + \delta \implies \frac{x^{3/2}}{(x-2)^2} > M$. We can immediately start putting $\frac{x^{3/2}}{(x-2)^2} > M$ under δ control by minimizing the numerator and maximizing the denominator:

$$\frac{x^{3/2}}{(x-2)^2} > \frac{2^{3/2}}{(2+\delta-2)^2} \ge M \tag{12}$$

$$\implies \frac{2^{3/2}}{\delta^2} \ge M \tag{13}$$

$$\implies \frac{\delta^2}{2^{3/2}} \le \frac{1}{M} \tag{14}$$

$$\implies \delta \le \frac{2^{3/4}}{\sqrt{M}} \tag{15}$$

For horizontal asymptotes as $x \to \infty$:

Theorem: If for every $\epsilon > 0$, a $A > 0$ can be found such that $x > A \implies |f(x) - L| < \epsilon$, then $\lim_{x \to \infty} = L$.

Example 3: (Lecture 15, Assigned) Prove the limit $\lim_{x\to\infty} \frac{1}{x^r} = 0$ where $r > 0$.

For any $\epsilon > 0$, we want to pick a $A > 0$ such that $x > A \implies \left|\frac{1}{x^r}\right| < \epsilon$. We can place the LHS of $\left|\frac{1}{x^r}\right| < \epsilon$ straight away by minimizing the denominator by selecting the lower bound of x, which is A to get:

$$\frac{1}{x^r} < \frac{1}{A^r} \leq \epsilon \implies A \geq \epsilon^{1/r} \tag{16}$$

so choosing $A = \epsilon^{1/r}$ will always work.

2 Limit Theorems

Here are the limit theorems covered in class. Given $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ are both well defined, then:

- **Constant Limit Theorem:** $\lim_{x\to c} A = A$
- **Additivity Limit Theorem:** $\lim_{x\to c} [f(x) + g(x)] = L + M$
- **Product Limit Theorem:** $\lim_{x\to c} [f(x)g(x)] = LM$
- **Polynomial Limit Theorem:** $\lim_{x\to c} P(x) = P(c)$ if $P(x)$ is a polynomial.
- **Rational Function Limit Theorem:** $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$
- **Root Limit Theorem:** $\lim_{x\to c} f(x)^{1/n} = L^{1/n}$
- **Sandwich Limit Theorem:** If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ and $f(x) \leq g(x) \leq h(x)$ near c but not necessarily at c, then $\lim_{x\to c} g(x) = L$.

To help with trigonometry problems, here are a few properties you should know (and understand how to derive):

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\sin x \leq x \leq \tan x$ for $x \geq 0$. Since all these functions are odd, the inequality works in reverse for $x < 0$.
- $\sqrt{1 - x^2} \leq \cos x \leq 1$

Tip: When solving difficult trigonometry limits, try to break it up into $\sin x/x$ terms. If not possible, try to either bound the limit using the sandwich limit theorem, or bash through applying trig identities.

3 Continuity Theorems

Here are the definitions for continuity at different points:

- **Continuity at a point**: $f(x)$ is continuous at c if $\lim_{x\to c} = f(c)$
- **Continuity on the right**: $f(x)$ is continuous on the right of c if $\lim_{x\to c^+} = f(c)$.
- **Continuity on the left**: $f(x)$ is continuous on the left of c if $\lim_{x\to c^-} = f(c)$.
- **Continuity on open interval**: $f(x)$ is continuous on (a, b) iff $f(x)$ is continuous at all $x \in (a, b)$.
- **Continuity on closed interval**: $f(x)$ is continuous on $[a, b]$ iff $f(x)$ is continuous at all $x \in (a, b)$ and $f(x)$ is continuous

from the right of a and from the left of b.

There are also a few continuity theorems discussed in class:

- Given f, g, is continuous at a, then $f(x) + g(x)$ is continuous at a.
- If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

4 Derivative Theorems

The derivative $f'(x)$ is defined as:

$$f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{17}$$

where h is a dummy variable. A few definitions:

- **Differentiability at a point:** If $f'(a)$ exists, we say that $f(x)$ is differentiable at a.
- **Differentiability of function:** If $f'(x)$ is differentiable at all $x \in$ domain of $f(x)$, then $f(x)$ is a differentiable function.
- **Differentiability on open interval:** $f(x)$ is differentiable on (a, b) if $f'(x)$ is defined for all $x \in (a, b)$
- **Differentiability on closed interval:** $f(x)$ is differentiable on $[a, b]$ if $f'(x)$ is defined for all $x \in (a, b)$ and the right hand derivative at a exists and the left hand derivative at b exists.
- **Relation to Continuity:** Given $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

When evaluating derivatives, there are a few theorems that we've learned. The following only apply if the derivatives of each function exists.

- **Constant DT:** If $f(x) = C$, then $f'(x) = 0$.
- **Additivity DT:** $(f + g)' = f' + g'$
- **Product DT:** $(fg)' = f'g + fg'$
- **Power DT:** If $f(x) = Cx^n$, then $f'(x) = nCx^{n-1}$.
- **Poly DT:** If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$, then $P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$.
- **Reciprocal DT:** $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$
- **Quotient DT:** $(f/g)' = \frac{f'g - fg'}{g^2}$.
- **Chain DT:** $\frac{d}{dx} f(g(x)) = g'(x) f'g(x) \iff \frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

5 Features of a Graph

We can look at extrema points with derivatives:

- **Absolute Max**: $f(x)$ has an absolute maximum at c if $f(c) \geq f(x)$ for all $x \in$ domain of $f(x)$.
- **Absolute Max in closed interval**: $f(x)$ has an absolute max on $[a, b]$ if $f(c) \geq f(x)$ for all $x \in [a, b]$.
- **Local Max**: $f(x)$ has a local max at c if $f(c) \geq f(x)$ for some open interval containing c.

Here are a few important theorems:

Theorem: **Intermediate Value Theorem:** Given that $f(x)$ is continuous on $[a, b]$ and C is some number such that $f(a) < G(a) < f(b)$, there exists some C in $[a, b]$ such that $f(C) = G$.

Theorem: **Extreme Value Theorem:** Given $f(x)$ is continuous on $[a, b]$, then $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(d)$ for some $c, d \in [a, b]$.

Theorem: **Rolle's Theorem**: Given that f is continuous on $[a, b]$ and f is differentiable on $[a, b)$ and $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$. Note that there may be more than one c.

Theorem: **Mean Value Theorem:** Given that $f(x)$ is continuous on $[a, b]$ and $f(x)$ is differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

5.1 Estimation

We can approximate a function $f(x+\Delta x)$ as: $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$. For example, this allows us to estimate something like $29^{1/3}$ as $27^{1/3} + \left.\dfrac{d}{dx}x^{1/3}\right|_{x=27} \cdot 2$.

An approximation by itself is useless without a bound. We can create lower and upper bounds by applying the MVT between $[x, x + \Delta x]$ and/or between $[x + \Delta x, x_1]$ and finding the minimum and maximum values for $f'(x)$.

5.2 Curve Sketching Prelims

We can use Fermat's theorem to determine critical points:

Definition: c is a critical point of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

Here are some key features that might be seen on a graph:

- **Concavity:** If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I. If it lies below, then it is concave down.
- **Cusp:** A point c is a cusp if $f(x)$ is continuous at $x = c$ but $\lim_{x\to c^-} f(x) = \pm\infty$ and $\lim_{x\to c^+} f(x) = \mp\infty$.
- **Vertical Tangent:** A vertical tangent occurs when $\lim_{x\to c} |f'(x)| = \infty$ and $f(x)$ is continuous at c.
- **Slant Asymptote:** If $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$, then $y = mx + b$ is a slant asymptote to $f(x)$ at $+\infty$.
- **Inflection point:** A point of inflection is at c if $f(x)$ is continuous at c and the sign of concavity changes at c.

A function is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I. Although we can use this definition to find local max/mins, there are a few cutie (QT/quick test) ways to do so:

- **QT1: Increasing/Decreasing Test.** If f is differentiable on the interval I, we show that if $f' > 0$, f is increasing. If $f' < 0$, f is decreasing. If $f' = 0$, f is constant.
- **QT2: First Derivative Test** Given that I contains a critical point and f is continuous at c_{crit}, and f is differentiable in I but not necessarily at c_{crit}. Then, if $f' > 0$ to the left of c_{crit} and $f' < 0$ to the right, then c_{crit} is a local max. If it's the opposite, we get the local minimum.
- **QT3: Concavity** Given that $f(x)$ is twice differentiable on I, then $f''(x)$ exists on I. As a result if $f''(x) > 0$, f is concave up. If $f'' < 0$, f is concave down.

- **QT4: Second Derivative Test** Given that $f''(x)$ is continuous near c and $f'(c) = 0$, then if $f''(c) > 0$, $f(c)$ is a local minimum. If $f''(c) < 0$, $f(c)$ is a local maximum. If $f''(c) = 0$, there is no verdict.

In general, the recipe to test for local max and min is to:

- Find all c_{crit}.
- If QT4 applies, use it.
- If it doesn't, and if QT2 applies, use it.
- If QT2 doesn't apply, use the basic definition of increasing/decreasing.

5.3 Curve Sketching Steps

1. Determine general behaviour:
   - Find Domain / Range / Limits at ∞.
   - Determine endpoints if they exist.
   - Find vertical, horizontal, slant asymptotes if they exist:
2. Determine x and y intercepts.
3. Establish if $f(x)$ is symmetrical, even, odd, and/or periodic.
4. Find $f'(x)$ and use this to:
   - Find all critical points and $f(c_{\text{crit}})$.
   - Find when $f(x)$ is increasing/decreasing.
   - Apply QT2.
   - Find vertical tangents / cusps if they exist.
5. Find $f''(x)$ and use it to:
   - Find when $f(x)$ is concave up/down.
   - Find points of inflection if they exist.
   - Optional: Use QT4 to confirm local max/min
6. Determine the absolute maximum and min by choosing the largest and smallest values of f, if they exist.