ESC194: Midterm 2 Review

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Please let me know via discord (Qcumber#4444) if I am missing anything, there exists any typos, and especially if something is horrendously wrong! Note that this is an unofficial resource and I am not responsible if the use of this study sheet causes you to fail the midterm, break up with your partner, find your house burned down, or be captured by the North Korean government to be forced to work on their nuclear missile project which leads to the destruction of the entire world.

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1 Curve sketching

1.1 Formally Defining Horizontal Aymptotes

Horizontal asymptotes are formally defined as:

> **Definition**: A horizontal asymptote occurs when $\lim\limits_{x \to \infty} f(x) = L$. We can say that $f(x)$ goes to L as x goes to infinity if for any $\epsilon > 0$, a number A can be found s.t. for all $x > A$, $|f(x) - L| < \epsilon$.
>
> The idea behind this revolves around finding f values as close to L as might be wanted by going to large enough x values.

An important theorem to determine horizontal asymptotes of reciprocal functions:

> **Theorem**: The reciprocal horizontal asymptote limit:
>
> $$\lim_{x \to \pm\infty} \frac{1}{x^r} = 0 \tag{1}$$

1.2 Prelims

We can use Fermat's theorem to determine critical points:

> **Definition**: c is a critical point of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

Here are some key features that might be seen on a graph:

- **Concavity:** If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I. If it lies below, then it is concave down.

- **Cusp:** A point c is a cusp if $f(x)$ is continuous at $x = c$ but $\lim\limits_{x \to c^-} f(x) = \pm\infty$ and $\lim\limits_{x \to c^+} f(x) = \mp\infty$.

- **Vertical Tangent:** A vertical tangent occurs when $\lim\limits_{x \to c} |f'(x)| = \infty$ and $f(x)$ is continuous at c.

- **Slant Asymptote:** If $\lim\limits_{x \to \infty} [f(x) - (mx + b)] = 0$, then $y = mx + b$ is a slant asymptote to $f(x)$ at $+\infty$.

- **Inflection point:** A point of inflection is at c if $f(x)$ is continuous at c and the sign of concavity changes at c.

A function is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I. Although we can use this definition to find local max/mins, there are a few cutie (QT/quick test) ways to do so:

- **QT1: Increasing/Decreasing Test.** If f is differentiable on the interval I, we show that if $f' > 0$, f is increasing. If $f' < 0$, f is decreasing. If $f' = 0$, f is constant.

- **QT2: First Derivative Test** Given that I contains a critical point and f is continuous at c_{crit}, and f is differentiable in I but not necessarily at c_{crit}. Then, if $f' > 0$ to the left of c_{crit} and $f' < 0$ to the right, then c_{crit} is a local max. If it's the opposite, we get the local minimum.

- **QT3: Concavity** Given that $f(x)$ is twice differentiable on I, then $f''(x)$ exists on I. As a result if $f''(x) > 0$, f is concave up. If $f'' < 0$, f is concave down.

- **QT4: Second Derivative Test** Given that $f''(x)$ is continuous near c and $f'(c) = 0$, then if $f''(c) > 0$, $f(c)$ is a local minimum. If $f''(c) < 0$, $f(c)$ is a local maximum. If $f''(c) = 0$, there is no verdict.

In general, the recipe to test for local max and min is to:

- Find all c_{crit}.

- If QT4 applies, use it.

- If it doesn't, and if QT2 applies, use it.

- If QT2 doesn't apply, use the basic definition of increasing/decreasing.

1.3 Curve Sketching Steps

1. Determine general behaviour:

   - Find Domain / Range / Limits at ∞.

   - Determine endpoints if they exist.

   - Find vertical, horizontal, slant asymptotes if they exist:

2. Determine x and y intercepts.

3. Establish if $f(x)$ is symmetrical, even, odd, and/or periodic.

4. Find $f'(x)$ and use this to:

   - Find all critical points and $f(c_{\text{crit}})$.

   - Find when $f(x)$ is increasing/decreasing.

   - Apply QT2.

   - Find vertical tangents / cusps if they exist.

5. Find $f''(x)$ and use it to:

   - Find when $f(x)$ is concave up/down.

   - Find points of inflection if they exist.

   - Optional: Use QT4 to confirm local max/min

6. Determine the absolute maximum and min by choosing the largest and smallest values of f, if they exist.

2 Optimization

Here is a checklist for solving optimization problems. If we want to optimize f:

- Check critical points.

- Check for endpoints.

- Check for local max, min.

- Check $\lim\limits_{x \to \infty}$ and $\lim\limits_{x \to -\infty}$.

- Make a decision.

2.1 Numerical Methods

> **Theorem**: The **method of successive bisections** can be performed if f is a continuous function and we can find values a and b such that $f(b) < 0 < f(a)$. These two values can be determined by trial and error. By IVT, the root must exist in between a and b. To use this method, we calculate the halfway point x_{h1}. If $f(x_{h1})$ is positive, we replace a with x_{h1}. If it's negative, we replace b with x_{h1}.

Theorem: Using **Newton's Method** is much faster computationally. However, there is the added restriction that $f(x)$ *must* be differentiable. It works in the following steps:
1. Make a first guess for the root, x_1
2. Find the equation for the tangent line at $(x_1, f(x_1))$
3. Find the x intercept of the tangent line, and let

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

and continue with x_2. Note however, that this doesn't always work such as when it diverges away from the root such as $x^{1/3}$.

Here are the overall steps that are recommended:

1. Try Newton's Method first if function is differentiable.

2. If the x_n values converge, great!

3. If they do not, try another value.

4. If they still diverge, use the method of successive bisections.

3   Formal Definition of an Integral

The summation notation is denoted below:

Definition: If $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \tag{3}$$

There are a few theorems:

- For a constant α:
$$\sum_{i=m}^{n} \alpha a_i = \alpha \sum_{i=m}^{n} a_i \tag{4}$$

- It is also linear:
$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \tag{5}$$

- $\sum_{i=1}^{n} \alpha = \alpha n$

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2} \right)^2$

- $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$

One way of defining an integral is thinking of the area under the curve. This introduces the concept of a **Riemann Sum**:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{6}$$

where x_i represents points in the **partition** of the domain in which we want to approximate the area. The approximation gets more and more precise at the size Δx_i decreases. A few technical definitions to help:

> **Definition**: A **partition** is a finite subset of the closed interval $[a, b]$, which contains the points a and b. Denoted by P.

> **Definition**: The **norm** of $P = \|P\|$ which is the length of the longest subinterval:
>
> $$\|P\| = \max\left(\Delta x_1, \Delta x_2, \ldots, \Delta x_n\right) \tag{7}$$

Which can all be tied together to formally define the definite integral.

> **Definition**: If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition $x_0, x_1, x_2, \ldots, x_n$ where:
> $$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{8}$$
> Choose points x_i^* within each subinterval $[x_i + 1, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral** of f from a to b is:
> $$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{9}$$
>
> if the limit exists. If the limit does exist, then f is called integrable on the interval $[a, b]$. The sign $\int$ is called the integral sign. $f(x)$ is known as the **integrand**, and a, b are the limits of integration. The output is a single number that does not depend on x.

We can formally show that the definite integral can take on a specific value I with a $\delta - \epsilon$ statement:

> **Idea**: If we have:
> $$\int_a^b f(x)\,\mathrm{d}x = I \tag{10}$$
> then for ever $\epsilon > 0$, there exists a $\delta > 0$ such that:
>
> $$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \epsilon \tag{11}$$
>
> for all partitions P of $[a, b]$ with $\|P\| < \delta$ and all possible choices of x_i^* in $[x_{i-1}, x_i]$.

However, going through this proof would be a nightmare. Instead, we can show **integrability** via the following theorem:

> **Theorem**: Continuous and/or piecewise continuous on $[a, b]$ guarantees integrability on $[a, b]$,

> **Definition**: A function is **piecewise continuous** if it only has a finite number of jump discontinuities.

Now that we know when the integral exists, we can find ways of calculating it from scratch:

> **Idea**: Going through with a full Riemann sum calculation is also tedious. As a result, here are a few conventions to make it easier:

- We usually select regular partitions:

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{n} \tag{12}$$

- And we select x_i^* to be the RH end point such that:

$$x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n} \tag{13}$$

Therefore, the integral can be written as:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \tag{14}$$

4 Properties of Integration

4.1 Definite Integral Properties

There are a few properties:

- Constant:

$$\int_a^b c\,\mathrm{d}x = c(b-a) \tag{15}$$

- Additivity:

$$\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x \tag{16}$$

- Constant Multiple:

$$\int_a^b c(f)x\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \tag{17}$$

- Changing Limits:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^z f(x)\,\mathrm{d}x + \int_z^b f(x)\,\mathrm{d}x \tag{18}$$

There are also **order properties** of integrals. If $a < b$, then:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,\mathrm{d}x \geq 0 \tag{19}$$

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then:

$$\int_a^b f\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x \tag{20}$$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then:

$$m(b-a) \leq \int_a^b f\,\mathrm{d}x \leq M(b-a) \tag{21}$$

- Absolute values:

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x \tag{22}$$

4.2   Fundamental Theorem of Calculus

The **first fundamental theorem of calculus** states that:

> **Theorem**: Let f be continuous on $[a, b]$. The function F is defined on $[a, b]$ by:
>
> $$F(x) = \int_a^x f(t)\, \mathrm{d}t \tag{23}$$
>
> is continuous on $[a, b]$, differentiable on (a, b), and has derivative:
>
> $$F'(x) = f(x) \tag{24}$$
>
> for all $x \in (a, b)$.

Rarely (never) will you get a simple question like this. Sometimes, the upper bound is a function $g(x)$ instead. If this is the case, then:

> **Idea**: Assuming that f is continuous in $[a, b]$, then the function F is defined on $[a, b]$ by:
>
> $$F(x) = \int_a^{g(x)} f(t)\, \mathrm{d}t \tag{25}$$
>
> has a derivative:
>
> $$F'(x) = g'(x)f(g(x)) \tag{26}$$
>
> for $x \in (a, b)$. To see why this is true, we can apply the chain rule:
>
> $$F'(x) = \frac{d}{dx} f(g(x)) = g'(x)f(g(x)) \tag{27}$$

The **second fundamental theorem of calculus** states that:

> **Theorem**: Let f be continuous on $[a, b]$. If G is any antiderivative for f on $[a, b]$, then:
>
> $$\int_a^b f(t)\, \mathrm{d}t = G(b) - G(a) \tag{28}$$

This can alternatively be written as:

$$\int_a^b F'(x)\, \mathrm{d}x = F(b) - F(a) \tag{29}$$

which can be interpreted as the net change of $F(x)$. For example:

$$\Delta x = \int_a^b v(t)\, \mathrm{d}t \tag{30}$$

gives the displacement from $t = a$ to $t = b$. The proofs for these two theorems are provided below:

> **Proof: (1st theorem)** For x and $x + h$ in (a, b),
>
> $$F(x + h) - F(x) = \int_a^{x+h} f(x)\, \mathrm{d}t - \int_a^x f(x)\, \mathrm{d}t \tag{31}$$
>
> $$= \int_a^x f(t)\, \mathrm{d}t + \int_x^{x+h} f(t)\, \mathrm{d}t - \int_a^x f(t)\, \mathrm{d}t \tag{32}$$
>
> $$= \int_x^{x+h} f(t)\, \mathrm{d}t \tag{33}$$

For $h \neq 0$, we have:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt \tag{34}$$

We can separate it into cases. If $h > 0$, then we can write per the extreme value theorem the minimum value of f as $f(u) = m$ and the maximum value as $f(v) = M$ for $u, v \in [x, x+h]$ such that:

$$mh \leq \int_x^{x+h} f(t) \, dt \leq Mh \tag{35}$$

or:

$$f(u)h \leq \int_x^{x+h} f(t) \, dt \leq f(v)h \tag{36}$$

which we can rewrite, after dividing through by h:

$$f(u) \leq \frac{F(x+h) - F(x)}{h} \leq f(v) \tag{37}$$

As $h \to 0$, we have $u \to x$ and $v \to x$. Therefore:

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \tag{38}$$

$$\lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x) \tag{39}$$

which gives us:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \tag{40}$$

or:

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x) \tag{41}$$

Proof: (2nd theorem) Given that $F(x) = \int_a^x f(t) \, dt$ is an antiderivative of f and given that G is an antiderivative, then:

$$F'(x) = G'(x) \implies F(x) = G(x) + C \tag{42}$$

We know that $F(a) = 0$, so $G(a) + C = 0$ or $C = -G(a)$, which gives:

$$\int_a^b f(t) \, dt = F(b) = G(b) - G(a) \tag{43}$$

4.3 Integration Tricks

The **u-substitution** essentially reverses the chain rule.

Idea: Suppose we have an integral in the form:

$$\int f(g(x)) g'(x) \, dx \tag{44}$$

If we let $u = g(x)$, then $du = g'(x) dx$. So we can simplify the integral to:

$$\int f(u) \, du = F(u) + C = F(g(x)) + C \tag{45}$$

Once we have the indefinite integral, we can use back substitution to find the definite integral. We can avoid this step using a change of variables.

Theorem:

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \tag{46}$$

In general, here are a few tips, in no particular order:

- Refer to the table of integrals at the back of the book. You are allowed to use them.

- Look for symmetry and periodicity.

- Draw a picture. Sometimes, you can avoid a complicated integral and use plain old geometry this way!

- For u-substitution, look for derivative-function pairs.

- If there are not too many terms, you can sometimes expand functions into a polynomial.

- Check if the integral even exists!

- Apply the first theorem of calculus, if applicable.

- See if the integral (or a similar one) is in the book.

5 Areas and Volumes

5.1 Areas Between Curves

Suppose we wish to find the **area between two curves** $f(x)$ and $g(x)$. We can do this by partitioning the area into infinitesimally small rectangles:

$$\Delta A_i = [f(x_i^*) - g(x_i^*)]\Delta x_i \tag{47}$$

so that the area is given by:

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)]\,\Delta x_i \tag{48}$$

$$= \int_a^b [f(x) - g(x)]\,\mathrm{d}x \tag{49}$$

If we let $f(x) \geq g(x)$ when $x \in [a, b]$, then this gives the difference in their areas $A_1 - A_2$. If the condition $f(x) \geq g(x)$ is not satisfied, then we must break up the integral into multiple parts (if we interpret the area as having a positive area only). We can modify the area formula to be:

$$A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x \tag{50}$$

Suppose we have a curve $x = f(y)$ and $x = g(y)$ instead. The area between $y = a$ and $y = b$ works in the same way:

$$A = \int_a^b |f(y) - g(y)|\,\mathrm{d}y \tag{51}$$

5.2 Volumes

We can determine the **volume** of a solid by partitioning it into thin cylinders whose axes area parallel to the x axis. Then we can break up the volume into thin sections:

$$V_i \simeq A_i\Delta x_i \tag{52}$$

so the volume is:

$$V = \int_a^b A(x)\,\mathrm{d}x \tag{53}$$

which is the general formula for the volume of any shape. If we can figure out $A(x)$ and the necessary bounds, we can find the volume forany change.

Idea: For **solids of revolution**, we rotate a curve $f(x)$ about the x axis. The volume of this solid using the **disk method** is:

$$V = \int_a^b \pi f(x)^2 \, \mathrm{d}x \tag{54}$$

Similarly around the y axis:

$$V = \int_c^d \pi g(y)^2 \, \mathrm{d}x \tag{55}$$

For the volume by rotating the region between two curves $f(x)$ and $g(x)$, we get:

$$V = \int_a^b \pi(f(x)^2 - g(x)^2) \, \mathrm{d}x \tag{56}$$

which is known as the **washer method**.

Sometimes, the disk and washer method is too difficult to apply.

Idea: We can use the **shell method about the y-axis** to find the volume when a curve is rotated about the y axis. Suppose we wish to rotate a curve $f(x)$ from $x = a$ to $x = b$ around the y axis. Then the volume is:

$$V = \int_a^b 2\pi x f(x) \, \mathrm{d}x \tag{57}$$

Similarly, if a curve is rotated about the x axis, we use the **shell method about the x-axis**:

$$V = \int_a^b 2\pi y f(y) \, \mathrm{d}y \tag{58}$$

6 Misc

I honestly don't know where this section belongs, so I'm just copying and pasting from my notes (which I actually spent a decent amount of effort on):

- The average of a discrete set $\{a_1, a_2, \ldots, a_N\}$ is given by:

$$a_{\mathsf{avg}} = \frac{1}{N} \sum_i^N a_i \tag{59}$$

- For a continuous distribution, we can extend this to:

$$f_{\mathsf{avg}} = \frac{1}{N} \sum_i^N f(x_i^*) \tag{60}$$

Taking the limit as $N \to \infty$, we get:

$$f_{\mathsf{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \tag{61}$$

Theorem: **Mean Value Theorem for Integrals**: If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that:

$$f(c) = f_{\mathsf{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \tag{62}$$

Proof: Define $F(x) = \int_a^x f(t)\,dt$. If we apply the mean value theorem to F, then:

$$F'(c) = \frac{F(b) - F(a)}{b - a} \tag{63}$$

for some $c \in [a, b]$. Now since:

$$F'(x) = f(x) \tag{64}$$

it becomes apparent that:

$$f(c) = \frac{\int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt}}{b - a} = \frac{1}{b - a}\int_a^b f(t)\,dt \tag{65}$$

- We can also introduce **inverse functions**.

> **Definition**: A function $f(x)$ is said to be one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Alternatively, we can say that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

- We can use the **horizontal line test**. If any horizontal line crosses the function more than one time, then it is not one-to-one.

> **Definition**: Let f be a 1-1 function with domain A and range B. Then its inverse function, f^{-1} has domain B and range A, and is defined by:
> $$f^{-1}(x) = x \iff f(x) = y \tag{66}$$
> Therefore:
> $$f^{-1}(f(x)) = f(f^{-1}(x)) = x \tag{67}$$
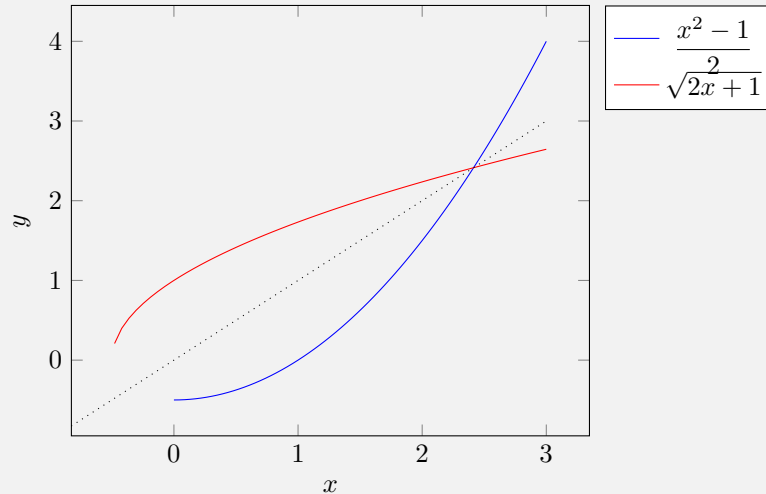
> **Warning**: To prevent confusion, not that:
>
> $$\frac{1}{f(x)} = [f(x)]^{-1} \neq f^{-1}(x) \tag{68}$$

- Geometrically, the inverse of a function represents a reflection of each point across the line $y = x$.

> **Example 1:** If $g(x) = \sqrt{2x + 1}$, it is implied that $x \geq -1/2$, so it is a one-to-one function. Therefore, the inverse function is:
> $$g^{-1}(x) = \frac{x^2 - 1}{2} \tag{69}$$

Inverse Function Example

Theorem: If f is either an increasing or decreasing function, then f is $1-1$, and hence, has an inverse.

Proof. Say $f(x)$ is decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$ and if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. □

Theorem: Let f be a 1-1 function defined on an interval I. If f is continuous, then f^{-1} is also continuous. (Proof provided in Appendix F)

- Let $g(x) = f^{-1}(x)$. Then:

$$f(g(x)) = x \tag{70}$$

$$\frac{d}{dx}f(g(x)) = 1 \tag{71}$$

$$f'(g(x))g'(x) = 1 \tag{72}$$

$$g'(x) = \frac{1}{f'(g(x))} \tag{73}$$

or:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{74}$$

which is equivalent to:

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} \tag{75}$$

Theorem: The inverse of composite functions is given by:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \tag{76}$$

Proof. Let $y = (f \circ g)^{-1}(x)$. Then:

$$x = (f \circ g)(y) = f(g(y)) \tag{77}$$

so we have:

$$g(y) = f^{-1}(x) \tag{78}$$

$$y = g^{-1}(f^{-1})(x) \tag{79}$$

$$= (g^{-1} \circ f^{-1})(x) \tag{80}$$

□

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7 Logarithms and Exponentials

> **Warning**: Note that in this section, I make the assumption you are already familiar with general logarithm and exponential properties, so I won't spend time writing those down.

> **Definition**: A **logarithm function** is a nonconstant differentiable function f defined for $x \in \{\mathbb{R}, (0, \infty)\}$ such that for all $a > 0$ and $b > 0$:
> $$f(a \cdot b) = f(a) + f(b) \tag{81}$$
> It has the following properties:
> - $f(1) = 0$
> - $f(1/x) = -f(x)$
> - $f(x/y) = f(x) - f(y)$
> - $f'(x) = \dfrac{1}{x} f'(1)$.

This leads to the definition of the **natural logarithm**:

> **Definition**: The natural logarithm is defined as:
> $$\ln(x) = \int_1^x \frac{\mathrm{d}t}{t} \tag{82}$$

Note that $\ln(x)$ is not the antiderivative of $\dfrac{1}{t}$. We can instead write:

$$\int \frac{\mathrm{d}t}{dt} = \ln|x| + C \tag{83}$$

since x can be negative as well.

> **Theorem**: **Feynman's trick of Differentiation**[a] (otherwise known as logarithmic differention): The following was popularized by Richard Feynman during the first of his Feynman Lectures. If we have a function:
> $$g(x) = g_1(x)g_2(x)g_3(x)\cdots g_n(x) \tag{84}$$
> Then taking the natural logarithm of both sides, applying the chain rule, and simplifying gives:
> $$g'(x) = g(x)\left(\frac{g_1'}{g_1} + \frac{g_2'}{g_2} + \cdots + \frac{g_n'}{g_n}\right) \tag{85}$$
> ---
> [a]Note that this is not a formal name. I just chose it to name it after Feynman because I'm a huge Feynman stan and I first heard about it in the preface to the Feynman Lectures where he was talking about mathematical tricks.

Exponential functions can be introduced:

> **Definition**: If z is a real number, then e^z is the number such that:
> $$\ln(e^z) = z \tag{86}$$

More formally, we can write the exponential function as $\exp(x) = e^x$. The most useful property of e^x is that:

$$\frac{d}{dx}e^x = e^x \tag{87}$$

We can extend this to general logarithmic and exponential functions. If $x > 0$, then we can define:

> **Definition**: The **general exponential** function is defined as
>
> $$x^z = e^{z \ln x} \tag{88}$$
>
> if $x > 0$.

Similarly:

> **Definition**: The **general logarithm** can be defined as:
>
> $$\log_p(x) = \frac{\ln x}{\ln p} \tag{89}$$

such that:

$$\frac{d}{dx} a^x = \ln(a) a^x \tag{90}$$

and

$$\frac{d}{dx} \log_p(x) = \frac{1}{x \ln p} \tag{91}$$

7.1 Bounding e

> **Idea**: We can first bound e^x by setting a lower limit (which happens to be the Taylor series!). Notice that via integration:
>
> $$e^x = 1 + \int_0^x e^t \, \mathrm{d}t \tag{92}$$
>
> Since e^x is always increasing, we can claim that $e^x > 1$ for $x > 0$ such that:
>
> $$e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x 1 \, \mathrm{d}t = 1 + x \tag{93}$$
>
> We can then repeat the previous step to show that
>
> $$e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1+x) \, \mathrm{d}t = 1 + x + \frac{x^2}{2} \tag{94}$$
>
> Repeating the process, we eventually get:
>
> $$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{95}$$

Instead of choosing to bound e^x, we can also choose to bound e. We have that:

$$\ln x = \int_1^x \frac{\mathrm{d}t}{t} \tag{96}$$

such that:

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} < \int_1^{1+1/n} 1 \, \mathrm{d}t \tag{97}$$

Since $frac1t < \dfrac{1}{1}$ for $t > 0$. The upper bound then becomes:

$$1 + \frac{1}{n} - 1 = \frac{1}{n} \implies \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \tag{98}$$

We can similarly repeat this process:

$$1 + \frac{1}{n} < e^{1/n} \implies (1 + \frac{1}{n})^n < e \tag{99}$$

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Note that if we take the limit as $n \to \infty$, *intuitively* we would expect the upper bound to become closer and closer to the true value. We shall explore this further, and we can write the lower bound as:

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} > \int_1^{1+1/n} \frac{\mathrm{d}t}{1 + 1/n} \tag{100}$$

since $\dfrac{1}{t} > \dfrac{1}{1 + 1/n}$. We can write this in logarithm form to get:

$$\ln\left(1 + \frac{1}{n}\right) > \left(\frac{1}{1 + 1/n}\right)\left(1 + \frac{1}{n} - 1\right) = \frac{1}{n+1} \implies \left(1 + \frac{1}{n}\right)^{n+1} > e \tag{101}$$

Putting it altogether, we have the following statement:

> **Idea**: e can be estimated with its lower and upper bound with the following:
>
> $$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \tag{102}$$

8   Inverse Trigonometric Functions

We can define the inverse function of trigonometric functions by restricting their domain, such as from $-\pi/2$ to $\pi/2$ for $\sin(x)$.

> **Definition**: The inverse function for $\sin(x)$ is given by :
>
> $$\sin^{-1}(x) = \arcsin(x) \tag{103}$$

> **Warning**: You need to be very careful with the domain and range. Sometimes, if x falls out of the domain, it can lead to a different answer altogether, or it could be undefined.

There's a lot of formulas for this one, but to derive formula such as $\sin\left(\tan^{-1}(x)\right)$, you just need to draw a picture of a right angled triangle with one of the legs as x and either the hypotenuse or the other leg as 1. If proofs are not needed, there is a formula sheet with all properties at the end of the book.