ESC195 Midterm Review

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Contents

1 L'hopital's Rule

When the limit of a function takes the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can use L'hopital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \qquad \text{indeterminate} \tag{1}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{2}$$

Not all indeterminate forms take this form however. Below is a list of indeterminate forms and how we can transform it to the one we're familiar with:

Indeterminate form	Conditions	Transformation to $0/0$	Transformation to ∞/∞
$\frac{0}{0}$	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=0$	—	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
$\frac{\infty}{\infty}$	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	—
$0\cdot\infty$	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{g(x)}{1/f(x)}$
$\infty-\infty$	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}(f(x)-g(x))=\lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x))=\ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x)=0^+, \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x)=1,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$

2 Integrals

A high level overview of steps that one should take when solving any integral is given in the flowchart. Here are a few more tips:

- Look for symmetry. If the function is odd or even, that may come into handy!
- For rational functions, try adding and subtracting the same term to the numerator.
- You can sometimes remove rationals (i.e. square roots) with the substitution $x = u^2$.

2.1 Improper Integrals

An improper integral is defined as:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x \tag{3}$$

And from $-\infty$ to ∞, we have:

$$\int_{-\infty}^\infty = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x \tag{4}$$

Here is an important theorem to determine when an integral converges to diverges:

Theorem: Comparison Test: Let f, g be continuous functions and $0 \le f(x) \le g(x)$ where $x \in [a,\infty)$,.

- If $\int_a^\infty g\,\mathrm{d}x$ converges, so does $\int_a^\infty f(x)\,\mathrm{d}x$.
- If $\int_a^\infty f\,\mathrm{d}x$ diverges, so does $\int_a^\infty g(x)\,\mathrm{d}x$.

2.2 Applications

Here are some things you can do with integrals:

- The arclength of a curve is:

$$s = \int_a^b \sqrt{1+f'(x)^2}\,\mathrm{d}x \tag{5}$$

- The surface area of a surface of revolution is:

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x \tag{6}$$

- The force a fluid exerts on the flat *wall* of a container is:

$$F = \int_a^b \rho g x w(y)\,\mathrm{d}y \tag{7}$$

  where $w(y)$ is the width as a function of height y.

- The centroid of a curve is given by:

$$\bar{x} = \frac{\int_a^b x f(x)\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x} \tag{8}$$

  and

$$\bar{y} = \frac{\int_a^b f(x)^2\,\mathrm{d}x}{2\int_a^b f(x)\,\mathrm{d}x} \tag{9}$$

- Pappus's Centroid theorem can be used to easily find the volume of revolution:

$$V = 2\pi R A \tag{10}$$

  where R is the distance from the centroid to the axis of rotation and A is the area of the rotated regoin.

- Similarly, we can also extend this to the centroid, which tells us that the surface area of a surface of revolution is:

$$A = 2\pi R d \tag{11}$$

where d is the arclength of the curve.

3 Parametric Equations

Parametric equations are parametized by $x(t)$ and $y(t)$. As a result, they form a plane curve.

- The derivative of the function at time t is given by:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \tag{12}$$

- The area of a parametric curve is given by:

$$A = \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{13}$$

- For a closed loop, the area can be represented in two ways:

$$A = -\int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t = \int_{t_1}^{t_2} x(t)y'(t)\,\mathrm{d}t \tag{14}$$

where t_1 and t_2 correspond to the same location. By convention, the curve has positive area when traversed counterclockwise.

- The arclength can be written as:

$$s = \int_{\alpha}^{\beta} \sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{15}$$

- The surface area when the curve is revolved is given by:

$$A = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \tag{16}$$

3.1 Graphing

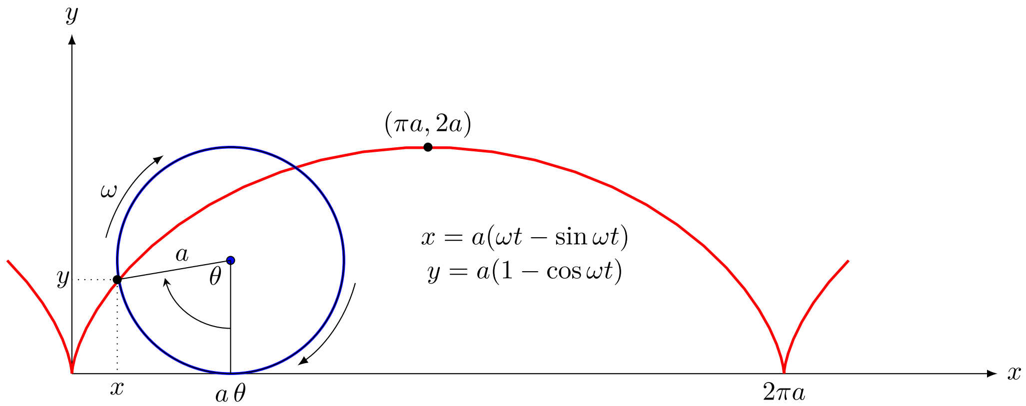
If possible (and if it is easy), convert the parametric equations to a cartesian equation and plot it normally. If the above will make the problem significantly harder, here is a checklist to plot a parametric function:

- Check for potential vertical tangents by setting $x'(t) = 0$.
- Check for potential horizontal tangents by setting $y'(t) = 0$.
- Find x and y intercepts by setting $x(t) = 0$ and $y(t) = 0$.
- Look for periodicity in either x, y, or both.
- Find the coordinate and the slope $\frac{dy}{dx}$ at $t = 0$ and at the endpoint $t = t_f$.
- Is $x(t)$ a $1-1$ function? If not, you'll get a graph where some points are directly above others.

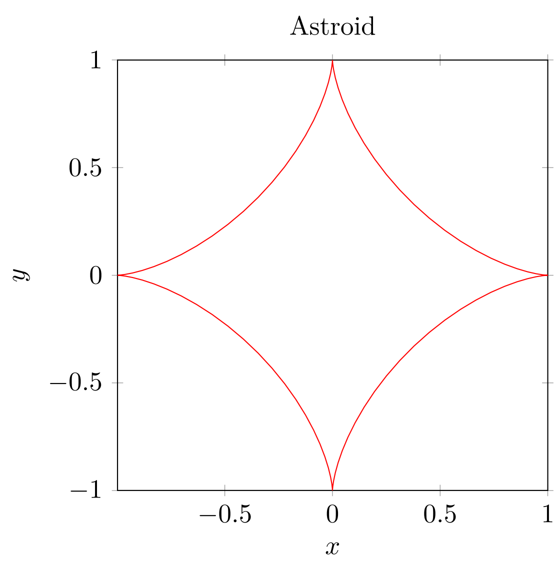
3.2 Common Parametric Curves

Here is a list of common parametric curves:

- A circle centered at (x_0, y_0) with radius r:
$$(x_0 + r\cos(\pm\omega t), y_0 + r\sin(\pm\omega t)) \tag{17}$$
where $\omega \in \mathbb{R}$.
- An ellipse centered at (x_0, y_0) with a horizontal length a and vertical length b:
$$(x_0 + a\cos(\pm\omega t), y_0 + b\sin(\pm\omega t)) \tag{18}$$
where $\omega \in \mathbb{R}$.
- A straight line with slope m passing through (x_0, y_0):
$$(x_0 + at, y_0 + amt) \tag{19}$$
where $a \in \mathbb{R}$.
- A **cycloid** (e.g. the curve traced by a point on a rolling circle with radius a with angular frequency ω) is given by:
$$(a\omega t - a\sin\omega t, a - a\cos\omega t) \tag{20}$$
which is seen in the below diagram:



- An **astroid** is given by:
$$(a\cos^3\omega t, a\sin^3\omega t) \tag{21}$$
which represents the curve traced out by a point on a rolling circle with radius $a/4$ rolling inside a circle of radius a.



4 Polar Curves

Polar curves are another type of plane curve defined using polar coordinates $[r, \theta]$. We can convert between cartesian and polar coordinates using the following transformations:

$$x = r\cos\theta \tag{22}$$
$$y = r\sin\theta \tag{23}$$
$$r = \sqrt{x^2+y^2} \tag{24}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{25}$$

Calculus related results are summarized below:

- The derivative is given by:
$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{26}$$
- The area is given by:
$$A = \frac{1}{2}\int_\alpha^\beta r(\theta)^2 \,\mathrm{d}\theta \tag{27}$$
- The arclength is given by:
$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{28}$$

4.1 Graphing

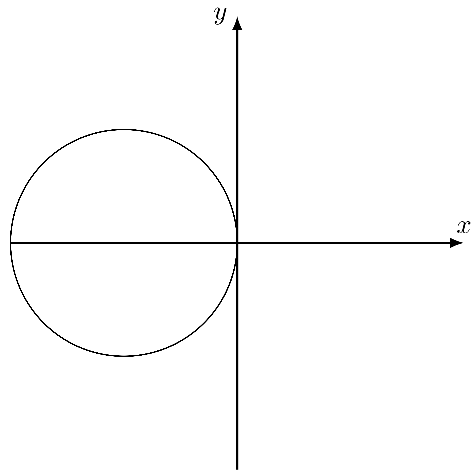
Here is a checklist for plotting polar graphs:

- Check for symmetry:
  - Symmetry about x axis: $r(\theta) = r(-\theta)$
  - Symmetry about y axis: $r(\pi - \theta) = r(\theta)$
  - Symmetry about origin: $r(\theta) = r(\theta + \pi)$
- Check the domain
- Find when $r = 0$
- Find local max and min values of r and where they are located, and break it up into intervals.
- It isn't always necessary but it may help to calculate $\frac{dy}{dx}$ at important locations.

4.2 Common Polar Curves

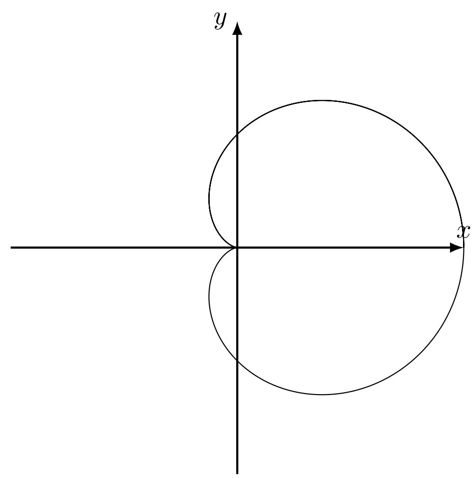
Note that the orientation of the below curves are not always fixed. It is possible to flip and rotate them by shifting the argument θ or using negative numbers. They just represent the general class of functions.

- Straight lines:
  - Straight lines $y = mx$: can be represented by $\theta = \arctan(m)$.
  - Vertical lines $x = a$: can be represented by $r = a\sec\theta$.
  - Horizontal lines $y = b$: can be represented by $r = b\csc\theta$.
- Circles:
$$r = -2\cos\theta \tag{29}$$



- **Cardioids**:

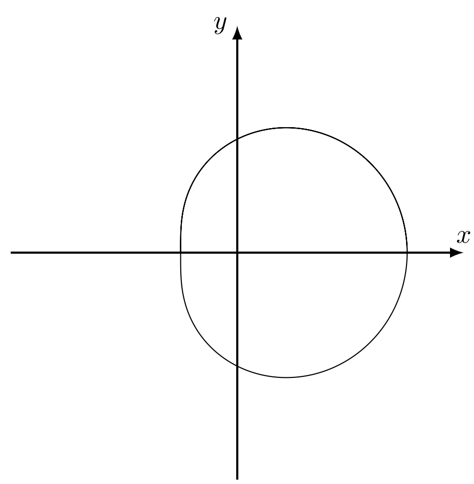
$$r = a + a\cos\theta \tag{30}$$



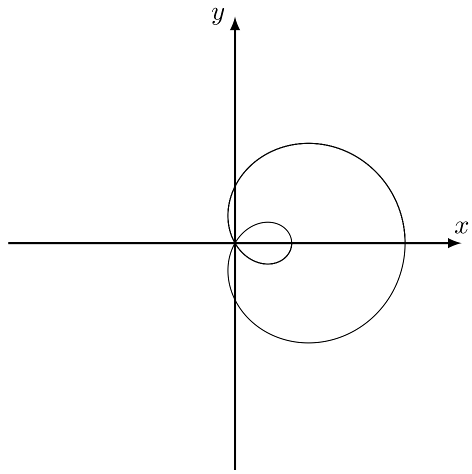
- **Limacons**:

$$r = a + b\sin\theta \tag{31}$$

There are two types, for $a > b$:

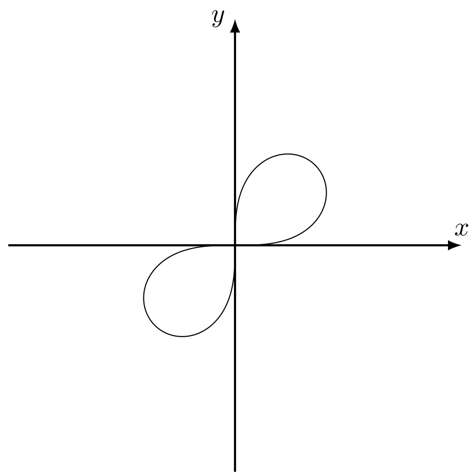


For $a < b$:



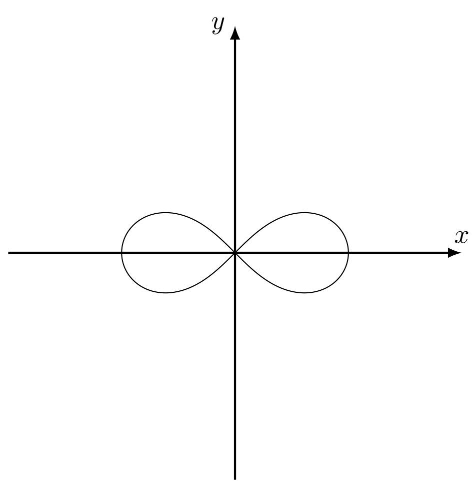
- **Leminiscates**. Again, there are two types. For:

$$r^2 = a\sin(2\theta) \tag{32}$$



and:

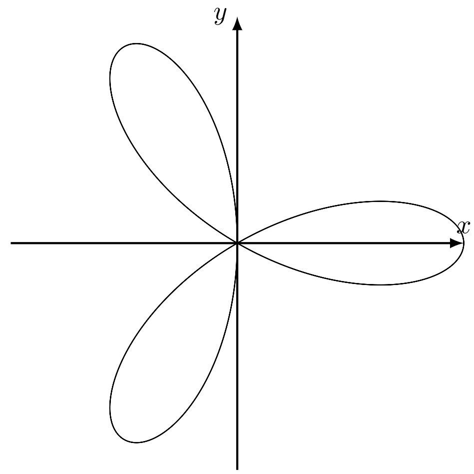
$$r^2 = a\cos(2\theta) \tag{33}$$



- **Petal curves**:

$$r = a\sin(n\theta) \tag{34}$$
$$r = a\cos(n\theta) \tag{35}$$

where n is an integer. There are n petals if n is odd and $2n$ petals if n is even. For example, the following is: $r = 2\cos 3\theta$:



and for $r = 2\sin(4\theta)$: