MAT185 Tutorial 1

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Note: The treatment of these tutorial questions are not always very rigorous. The general ideas however for a completely rigorous proof are provided and should not be difficult to complete.

1 Tutorial Problems

Problem One

(a) The vector space must be $\{\boldsymbol{x}\}$ where $\boldsymbol{x} = \boldsymbol{0}$. This is because the zero vector belongs in all vector spaces, and if this space only has one vector, then it must be the zero vector.

(b) We have:

$$\begin{aligned} c\boldsymbol{x} &= (c+d-d)\boldsymbol{x} && (1)\\ &= (c-d+d)\boldsymbol{x} && (2)\\ &= (c-d)\boldsymbol{x} + c\boldsymbol{x} \qquad (DSA) && (3)\\ & && (4) \end{aligned}$$

We have a vector $\boldsymbol{v} \equiv (c-d)\boldsymbol{x}$ such that:

$$\boldsymbol{v} + c\boldsymbol{x} = c\boldsymbol{x} \tag{5}$$

Per proposition 4, we must have $\boldsymbol{v} = \boldsymbol{0}$. Since $\boldsymbol{x}$ can be nonzero, then this means that $(c-d) = 0$ (Z) or $c = d$. Alternatively, we could arrive at this in a much easier way using the cancellation theorem.

(c) Proof by contradiction: Suppose for the sake of contradiction that a vector space V consists of N distinct vectors with $N > 0$. Per SC, if $\boldsymbol{x} \in V$, then $\lambda\boldsymbol{x} \in V$ with $\lambda \in \mathbb{R}$. However, there are infinite possible values of λ. We now need to show that $\lambda_1\boldsymbol{x} \neq \lambda_2\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{0}$ and $\lambda_1 \neq \lambda_2$. From (b), we have determined that if the two vectors are equal, then it must demand that $\lambda_1 = \lambda_2$, which isn't satisfied here, and thus we have found a contradiction. The vector space can however consist of one vector, the zero vector.

Problem Two

(i) Since we are using normal addition and scalar multiplication, then the addition and multiplication axioms are satisfied. We now need to show that this is closed. Let $x = \dfrac{a}{b}$ and $y = \dfrac{c}{d}$ with $b, d \neq 0$ and $\gcd(a,b), \gcd(b,d) \neq 0$. We need to show that their sum is also a rational number:

$$x + y = \frac{ad+cb}{bd} \tag{6}$$

and since it can be written as a fraction, $x+y$ is also rational. Note that from our earlier condition, $bd \neq 0$. Similarly for scalar multiplication by λ, we have:

$$\lambda x = \frac{\lambda a}{b} \tag{7}$$

which is also a fraction.

(ii) We first make sure that the space is closed under addition:

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} \tag{8}$$

Addition only changes the bottom left entry, and modifies it under normal addition rules. As a result, both addition closure and addition axioms will be satisfied since the set of all real numbers is a vector space. Similarly, scalar multiplication only affects the bottom left corner in the regular way, so this is a vector space.

(iii) So many things are violated here! Take associativity of addition for example:

$$(\boldsymbol{x}+\boldsymbol{x})+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{y} \tag{9}$$
$$= \boldsymbol{x} \tag{10}$$

but:

$$\boldsymbol{x}+(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{x}+\boldsymbol{y} \tag{11}$$
$$= \boldsymbol{y} \tag{12}$$

Problem Three

We first note that $a_1 = b_1 \equiv \lambda_1$ and $a_2 = b_2 \equiv \lambda_2$, otherwise commutativity does not hold. Also notice that the first index is independent from the second index during addition and scalar multiplication. As a result, we consider the simpler problem. Is the following a vector space?

$$\boldsymbol{x}+\boldsymbol{y} = a(x+y) \tag{13}$$

From associativity, we have:

$$(c+d)\boldsymbol{x} = c\boldsymbol{x}+d\boldsymbol{x} = \lambda_1(cx+dx) = \lambda_1(c+d)x \tag{14}$$

but also

$$(c+d)\boldsymbol{x} = (c+d)x \tag{15}$$

which implies that $\lambda_1 = \lambda_2 = a_1 = b_1 = a_2 = b_2 = 1$.

Problem Four

We only need to worry about closure since regular matrices are a vector space. We can tell that the vector space is closed under addition. Let $w(A)$ be the weight of a magic square $A \in \mathbb{M}_n$, then $w(A+B) = w(A)+w(B)$. Formally, let $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$. Then:

$$\boldsymbol{A}+\boldsymbol{B} = [a_{ij}+b_{ij}] \tag{16}$$

The sum of the m^{th} row of a magic square $\boldsymbol{A}$ is given by $S_m(\boldsymbol{A})$:

$$S_m(\boldsymbol{A}) = \sum_{k=1}^{n} a_{mk} \tag{17}$$

and thus:

$$S_m(\boldsymbol{A}+\boldsymbol{B}) = \sum_{k=1}^{n}(a_{mk}+b_{mk}) = \sum_{k=1}^{n} a_{mk} + \sum_{k=1}^{n} a_{nk} = S_m(\boldsymbol{A})+S_m(\boldsymbol{B}) \tag{18}$$

and similar reasoning for the columns and diagonals. We can also show that this is closed under scalar multiplication:

$$S_m(\lambda\boldsymbol{A}) = \sum_{k=1}^{n} \lambda a_{mk} = \lambda\sum_{k=1}^{n} a_{mk} = \lambda S_m(\boldsymbol{A}) \tag{19}$$

Remarks: It's interesting that this is almost closed under matrix multiplication, we can represent multiplication of two matrices by:

$$\boldsymbol{AB} = \left[\sum_{p=1}^{n} a_{ip}b_{pj}\right] \tag{20}$$

and thus:

$$S_m\left(\boldsymbol{AB}\right) = \sum_{k=1}^{n}\sum_{p=1}^{n} a_{mp}b_{pk} \tag{21}$$

We can interchange the sums:

$$\begin{aligned}
&= \sum_{p=1}^{n}\sum_{k=1}^{n} a_{mp}b_{pk} && (22)\\
&= \sum_{p=1}^{n}\left(a_{mp}\sum_{k=1}^{n} b_{pk}\right) && (23)\\
&= \sum_{p=1}^{n}\left(a_{mp}\cdot S_p(\boldsymbol{B})\right) && (24)\\
&= S_p(\boldsymbol{B})\cdot\sum_{p=1}^{n} a_{mp} && (25)\\
&= S_p(\boldsymbol{B})\cdot S_m(\boldsymbol{B}) && (26)
\end{aligned}$$

However for a magic square, $S_i = S_j$ for any $1 \leq i, j \leq n$ so this means that:

$$S_m(\boldsymbol{AB}) = S_m(\boldsymbol{A})S_m(\boldsymbol{B}) \tag{27}$$

In words, this means that for *any* row in the product $\boldsymbol{AB}$, the sum will equal to the product of the weights of the original two matrices. However, this result is not true for columns or diagonals.