MAT185 Tutorial 2

QiLin Xue

January 25, 2021

Note: The treatment of these tutorial questions are not always very rigorous. The general ideas however for a completely rigorous proof are provided and should not be difficult to complete.

1 Tutorial Problems

Problem One

(a) From the subspace test theorem, we must show three things:

(SI) We propose that the zero vector is $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which belongs in both U and W.

(SII) If $u_1, u_2 \in U$, then $u_1 + u_2 \in U$. This is because the top right entry will always be zero, and the matrix will always be in ${}^2\mathbb{R}^2$. Similar reasoning applies to V.

(SIII) The exact same reasoning applies as above.

As a result, U and W are both subspaces of ${}^2\mathbb{R}^2$.

(b) The intersection of U and W is:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

since the top right and bottom left diagonal entries must all be zero. The other entries are free to be anything in $\mathbb{R}$.

(c) Yes, per the same reasoning as (a).

Problem Two

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in U \cap V$. We need to show that:

(SI) If a vector $\mathbf{0}$ is the zero vector of both U and V, then it is also the zero vector of $U \cap V$.

(SII) We have $\boldsymbol{u} + \boldsymbol{v} \in U$ by the given statement and also $\boldsymbol{u} + \boldsymbol{v} \in V$. So by definition, $\boldsymbol{u} + \boldsymbol{v} \in U \cap V$.

(SIII) The exact same reasoning applies as above.

□

Problem Three

(a) We have:

$$U + W = \left\{ \begin{bmatrix} e & f \\ g & h \end{bmatrix} \middle| e, f, g, h \in \mathbb{R} \right\} = {}^2\mathbb{R}^2 \tag{1}$$

Proof. Let $\boldsymbol{u} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in U$ and $\boldsymbol{v} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in V$. Then: $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + x & y \\ b & y + c \end{bmatrix}$. Since all of these entries have a domain of $\mathbb{R}$ and are independent from each other, any matrix in ${}^2\mathbb{R}^2$ can be created. □

(b) Yes. Any vector space is a subspace of itself.

Problem Four

Proof. Let $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. We need to show that:

(SI) The zero vector is defined such that $0\boldsymbol{x} = \mathbf{0}$ where $x \in V$. This means that the zero vector will be the same for all subsets of V.

(SII) We can write each vector in $U + W$ as a composition of two vectors. For example, we have $\boldsymbol{u}_1 + \boldsymbol{w}_1 \in U + W$ by the given statement as well as $\boldsymbol{u}_2 + \boldsymbol{w}_2 \in U + W$. Therefore, the sum of these two vectors is:

$$\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{v}_1 + \boldsymbol{v}_2 \tag{2}$$

which is in $U + W$.

(SIII) Same reasoning as above.

$\square$

2 Unofficial Tutorial Problems

Problem One

Proof. To prove 185 is odd, let $k = 92$ such that $185 = 2k + 1$ and therefore it is odd by definition. To prove that -420 is odd, pick $k = -210$ such that $-420 = 2k$ and therefore it is even by definition. $\square$

Problem Two

If m, n are odd, we can write them in the form of $m = 2k_1 + 1$ and $n = 2k_2 + 1$ such that:

$$m + n = 2(k_1 + 1 + k_2) \tag{3}$$

If we pick $k = k_1 + 1 + k_2$, then $m + n$ is even by definition. Note: We can also prove this via modular arithmetic. We have:

$$m \equiv 1 \pmod 2$$
$$n \equiv 1 \pmod 2$$

Adding them, we get:

$$m + n \equiv 2 \pmod 2 \implies m + n \equiv 0 \pmod 0$$

Problem Three

Two integers m and n either have the parity, or they do not have the parity. We will show that if they have different parity, they will not be even. WLOG, let $m = 2k_1$ and let $n = 2k_2 + 1$ such that:

$$m + n = 2(k_1 + k_2) + 1$$

and we can choose $k_1 + k_2$ to be k to show that $m + n$ is odd. We also want to use the proposition that being odd or even is mutually exclusive.

Problem Four

For the sake of contradiction, assume that k divides $p! + 1$ (which can be written as $k \mid p! + 1$. This means that $\dfrac{p! + 1}{k}$ is an integer, or:

$$\frac{p(p-1)(p-2)\cdots(2)(1) + 1}{k} = \frac{p(p-1)(p-2)\cdots(2)(1)}{k} + \frac{1}{k}$$

Since $2 \leq k \leq p$, the first term is an integer. For the sum to also be an integer, the second term also needs to be an integer. However, $1/k$ is never an integer for $k \neq 1$, so our original assumption is false.

Problem Five

For the sake of contradiction, assume there are a finite number of primes. If so, multiply all the primes together and add one. By similar reasoning as in problem four, this cannot be divided by any number except 1 and itself, and thus is a new prime. Asa result, there are an infinite number of primes.

Problem Six

First, assume that a solution exists. If the solution exists, we will show that it must be in V.

Assume for the sake of contradiction that $\boldsymbol{x} \notin V$. Then, we have: $\boldsymbol{x} = \boldsymbol{v} + -\boldsymbol{u}$ from the cancellation theorem. From the closure axioms, then $\boldsymbol{x} \in V$. Now we show that the solution must exist. Again, this is trivial under the closure axiom.

3 Tutorial Worksheet

Task 2.1

The vector $(0, a, 0) \in Y$ and $(0, 0, b) \in Z$. Therefore, the intersection results in $(0, 0, 0)$ and $Y + Z = (0, a, b)$.

Task 2.2

Let $V = Y$ and $W = Z$. Then $(0, 3, 3) \in V + W$ but $(0, 3, 3) \notin V \cup W$.

Task 2.3

- This is the definition of the intersection.
- V is a subspace, so by definition $c\boldsymbol{s} \in V$ since $\boldsymbol{s} \in V$.
- Same reasoning as above.
- The vector $c\boldsymbol{s}$ is in both V and W so $c\boldsymbol{s} \in V \cap W$.
- See official problem set for how to finish.

Task 2.4

No, it's not always a subspace. See task 2.2.

Task 2.5

The subspace U represents a line with the same slope that passes through the origin and $\boldsymbol{p}$ represents the offset from the origin.

Task 2.6

Notice that the line L in the previous task is only not a suspace because the zero vector does not exist. We want a vector $\mathbf{0}$ such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ and the only zero vector that can do this is $\mathbf{0} = (0, 0)$. However, $0\boldsymbol{x} = \mathbf{0}$ is not necessarily contained in L.

Now, we can define the zero vector to be $\mathbf{0} = (\boldsymbol{p}, 0)$ such that $0\boldsymbol{x} = \mathbf{0}$ always and $(\boldsymbol{p}, 0) \in L$ (see the geometric interpretation of the previous task). Note that $\mathbb{R}^2_p$ is not a subspace of $\mathbb{R}^2$ because the rules for scalar multiplication and addition is different.