MAT292: Ordinary Differential Equations

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Fall 2021

Contents

1 Introduction

Covers 1.1: Mathematical Models and Solutions

- Big Idea: Differential equations model physical situations:
  - Take a physical situation and ODE-ify it (How do we model a cooling coffee cup?)
  - Understand an ODE without solving it (What can we deduce directly from $y' = y^2$?)
  - Study, categories, typecast ODEs and solve them

    Example 1: Suppose we have $y' = y/t + \ln t$ and $y' = y^2 + t$. Which of these are harder to solve (without actually solving them)?

    It turns out that the second one is harder as it is *non-linear*.

  - Handle ODEs numerically (What do we do when we cannot solve an ODE that models a real life phenomenon?)
  - The art of problem solving (How do I work with no strings attached?)
- What is a differential equation?

  Definition: A differential equation relates a function and its derivatives.

- We can understand ODEs without solving it:

  Example 2: Let's consider a cup of coffee in a room. We want to model its change in temperature over time. How do we do this?

  There are a lot of variables, so we have to simplify our model. The things we care about
  - The temperature of the coffee cup $y(t)$.
  - t is in minutes.
  - $y(t)$ is in Celsius.
  - The temperature in the room T (in Celsius).

  The things we ignore / simplify:
  - Temperature variation within the cup
  - Temperature variation in the room

  Exercise: Let's consider some suggestions for an ODE describing the temperature of a coffee cup in a room. Each of the following suggested ODEs contradicts our intuition in some way. How?
  - $y' = y^2$
    * T isn't in there
    * Temperature would always increase except if $y = 0$.
    * The hotter the coffee, the faster it heats up.
  - $y' = \dfrac{T}{y}$
    * If $T > 0, y > 0$, then $y' > 0$
    * The model doesn't work for coffee at $0°$C.
  - $y' = y[e^{y-T} + y^3]$
  - $y' = y - T$
  - $y' = T - y$
    * There should be a parameter that describes the physical properties (rate of heating/cooling will be different for different materials)

  Idea: Without solving an ODE, you can already make many predictions about its solution

  (and then, for example, judge your model)

- We introduce a few definitions

Definition: An **ordinary differential equation** (ODE) only considers a function of 1 variable and its derivatives

Definition: A **partial differential equation** considers a function of several variables and its derivatives.

- The most general ODE for a function $y(t)$ is:
  (a) $F[t, y, y'', \ldots, y^{(n)}]$ for $n \in \mathbb{N}$.
  (b) Any function that satisfies this equation is called *a solution*

Definition: The order of an ODE is the highest derivative of an ODE.

- An autonomous ODE is if the independent variable doesn't appear in the ODE.
- Systems of ODEs arise if we study several quantities depending on the same variable and how their changes interact.

Example 3: Assume that $p(t)$ and $o(t)$ describe the number of twitter followers of two accounts. If there is no interaction, what are reasonable ODEs for these two quantities?

$$p'(t) = kp(t) \tag{1}$$
$$o'(t) = \ell o(t) \tag{2}$$

Suppose that if in addition to "word of mouth," we consider the effects that these two tweets have, what are reasonable ODEs for the number of followers?

$$p'(t) = k \cdot p(t) - m \cdot o(t) \tag{3}$$
$$o'(t) = \ell \cdot o(t) - n \cdot p(t) \tag{4}$$

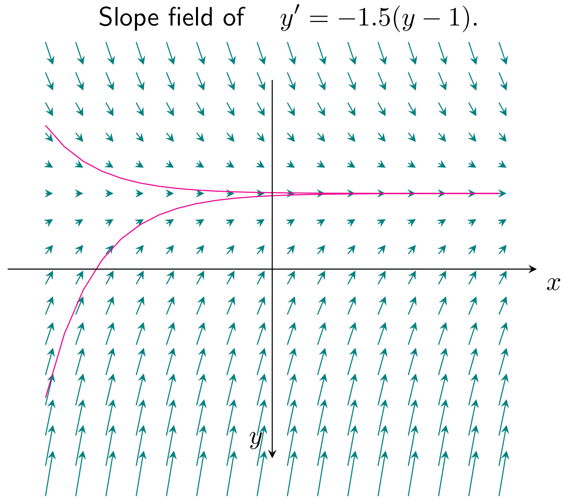
where all constant are positive. However, this is oversimplified as it assumes the people who follow *O* also follow *P*.

- Suppose a differential equation is given by

$$y'(t) = -1.5(y(t) - 1) \tag{5}$$

Definition: Consider the ODE $y' = f(t, y)$. We can draw a **direction field** as follows:
1. Draw a $t - y$ coordinate system.
2. Evaluate $f(t, y)$ over a rectangular grid of points.
3. Draw a line at each point (t, y) of the grid with slope $f(t, y)$.

Slope field of $y' = -1.5(y - 1)$.



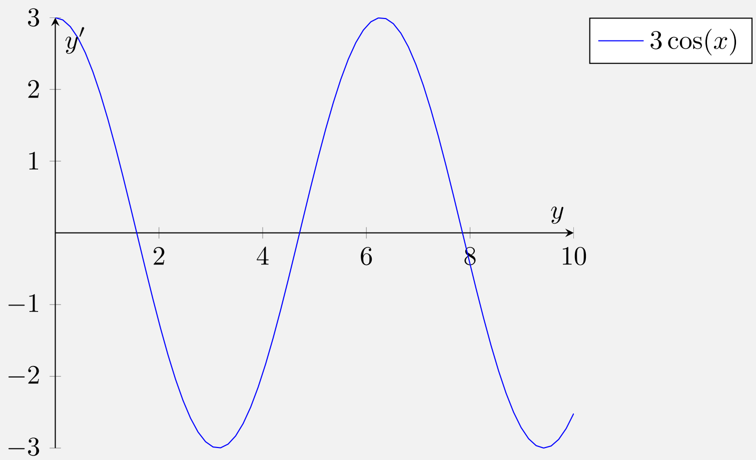
- Consider an **autonomous** first order ODE $y' = f(y)$. If $f(c) = 0$ for some specific value c, we call c an equilibrium of the ODE. We say it is

1. A **stable equilibrium,** if a solution starting at a value close to c approaches $y = c$ as $t \to \infty$
2. An **unstable equilibrium,** if a solution starting at a value close to c moves away from $y = c$ as $t \to \infty$.
3. A **semistable equilibrium,** if we observe either behaviour, depending on if the solution starts just above or just below c.

Warning: Stable, unstable, and semistable equilibrium are only well-defined for **autonomous** ODEs.

Example 4: Consider the differential equation $y' = 3\cos y$. We can construct the phas plot:

Phase Plot



Equilibrium occurs when $y' = 0$ The first equilibrium occurs at $y = \frac{\pi}{2}$. This is stable as if we move a bit to the left, y' is positive so that we move back to the right. If we move to the right instead, y' is negative and we move back to the left.

We can also determine this by looking at the second derivative $y'' = -3\sin(y)$. A negative second derivative means that it is stable. A positive second derivative means that it is unstable.

2 First Order ODEs

Note: This section will skip over separable ODEs

- **Linear Equations and the Integrating Factor**

Example 5: We want to find the general solution of $y' + 2ty = t$.

To do so, let's multiply the equation with $\mu = e^{t^2}$. Then:

$$e^{t^2}y' + e^{t^2}2ty = e^{t^2}t \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{t^2}y) = e^{t^2}t \tag{7}$$

$$e^{t^2}y = \int e^{t^2}t\,\mathrm{d}t \tag{8}$$

$$e^{t^2}y = \frac{1}{2}e^{t^2} + C \tag{9}$$

$$y = \frac{1}{2} + Ce^{-t^2} \tag{10}$$

where C depends on the initial value.

- The most general first order linear ODE is given by

$$a_0(t)y + a_1(t)y' = h(t), \tag{11}$$

which we can always turn into the form

$$y' + p(t)y = g(t) \tag{12}$$

if $a_1(t) \neq 0$ (if it was 0, then we can separate).

- We wish to find an integrating factor $\mu(t) > 0$, to solve $y' + p(t)y = g(t)$. We wish to multiply this by a factor of μ, to get

$$\mu y' + \mu p y = \mu y \iff \frac{\mathrm{d}}{\mathrm{d}t}(\quad \cdot \quad) = \mu g(t) \tag{13}$$

In order to write it like this, we want:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) = \mu y' + \mu' y \implies \mu' = \mu p. \tag{14}$$

We can solve this to get

$$\mu(t) = \exp\left(\int p(t)\,\mathrm{d}t\right), \tag{15}$$

and get the general solution to be

$$y = \frac{1}{\mu}\int \mu g\,\mathrm{d}t + \frac{C}{\mu} \tag{16}$$

Example 6: We want to solve $ty' + 2y = 4t^2$, $y(1) = 2$. We can rearrange it to

$$y' + \frac{2}{t}y = 4t. \tag{17}$$

The integrating factor is $\mu(t) = \exp\left(\int 2/t\,\mathrm{d}t\right) = t^2$. We can use this to solve

$$\mu y' + \mu\frac{2}{t}y = \mu 4t \iff t^2 y' + 2ty = 4t^3 \tag{18}$$

$$\iff (t^2 y)' = 4t^3 \tag{19}$$

$$\iff t^2 y = \int 4t^3\,\mathrm{d}t \tag{20}$$

$$\iff y(t) = t^2 + \frac{C}{t^2} \tag{21}$$

Using the initial value $y(1) = 2$, we get $\boxed{y = t^2 + \frac{1}{t^2}}$..

Note that we can't say anything about $y(-1)$. For example, the solution $t^2 + \frac{1}{t^2}$ for $t < 0$ is a *different* solution. Therefore, the particular solution is actually

$$y(t) = t^2 + \frac{1}{t^2} \quad t > 0. \tag{22}$$

3 The Initial Value Problem (IVP)

- How many initial conditions do we need, such that we only have one solution?

Theorem: Consider the IVP for the most general ODE $y' + p(t)y = g(t)$ with initial value $y(t_0) = y_0$ and an interval $I = (\alpha, \beta)$.

If:
  - $t_0 \in I$
  - $p(t)$ continuous on I
  - $g(t)$ continuous on I,

then this IVP has a solution and this solution is unique, and this solution exists for all $t \in I$.

Also, the ODE has a general solution that depends on only one constant C.

Example 7: Suppose we have the IVP

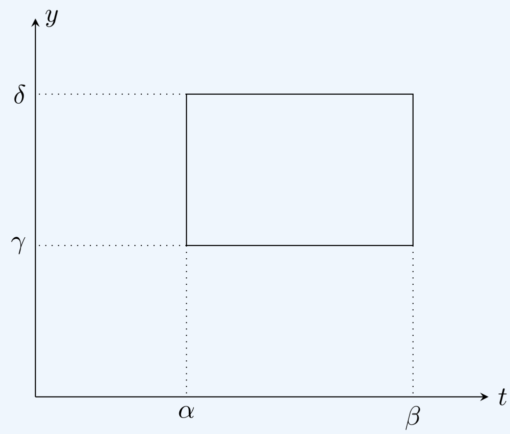
$$ty' + 2y' = 4t^2 \iff y' + 2\frac{y}{t} + 4t \tag{23}$$

with $y(1) = 2$ and let $t \neq 0$. By the above theorem, this has the **unique** solution

$$y = t^2 + \frac{1}{t^2} \tag{24}$$

Theorem: Consider the IVP $y' = f(t, y)$ and $y(t_0) = y_0$. Consider a rectangle $\alpha < t < \beta$, $\gamma < y < \delta$. If:

- the point (t_0, y_0) is in the rectangle:



- f is continuous in the rectangle
- f_y is continuous in the rectangle

Then the IVP has a unique solution. The solution exists for $\alpha < t < \beta$ for some interval $t_0 - h < t < t_0 + h$ where $h \neq 0$.

- Remarks:
  1. Non-linear ODEs don't necessarily have a general solution that depend on a single constant.
  2. The solution we get might be implicit, i.e. $\sqrt{y^2 + \ln(y)} = 5t$.

Example 8: Consider the ODE

$$(y + t^2 y)y' = 2t. \tag{25}$$

We can write

$$y' = f(t, y) = \frac{2t}{y + t^2 y} \tag{26}$$

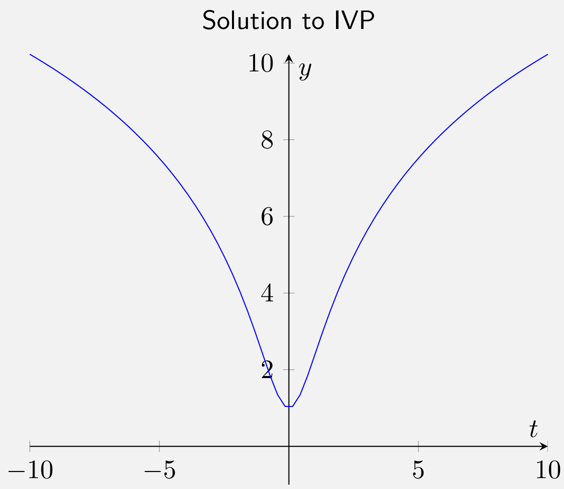
and

$$f_y(t, y) = -\frac{2t}{y^2 + y^2 t^2}.$$

The IVP is given by $f(0) = 1$. The rectangle for which y' and f_y is continuous is

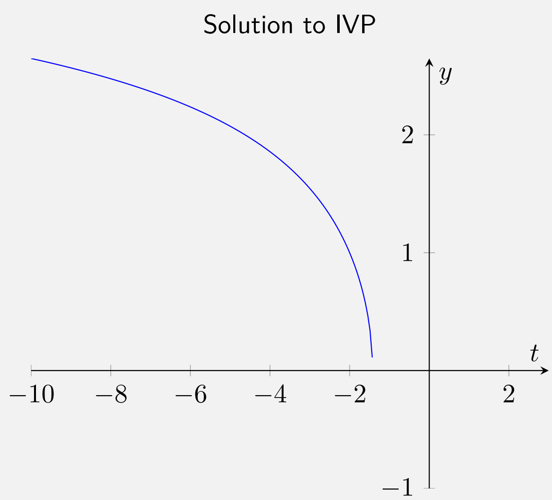
$$R = (-\infty, \infty) \times (0, \infty). \tag{27}$$

We can solve this by separation of variables and get the curve $y(t) = \sqrt{2\ln(t^2 + 1) + 1}$. We get



It turns out that the solution exists for all t, but we could not predict this!

Example 9: Now consider the same ODE but with the initial value $y(-2) = 1$. The solution is $y(t) = \sqrt{2\ln\left(\frac{t^2+1}{5}\right)+1}$, then the solution is in the following interval:



If instead the initial condition was $y(0) = 0$, note that we cannot surround the box such that f and f_Y is not continuous in that rectangle (i.e. at $y = 0$).

Warning: Note that the $E - U$ theorem is not an if and only if statement, i.e.

$$\text{condition fulfilled} \implies \text{solution exists} \tag{28}$$

but

$$\text{solution exists} \not\Longrightarrow \text{condition fulfilled} \tag{29}$$

- Some clarifications about the Picard–Lindelöf (E & U) theorem:
  - We need $f(t, y)$ continuous in the rectangle to guarantee existence.
  - We need $f_y(t, y)$ continuous in the rectangle to get uniqueness.
- There are no general solution for nonlinear ODEs, for example, take $y'y =^2$, then using separation of variables, we get

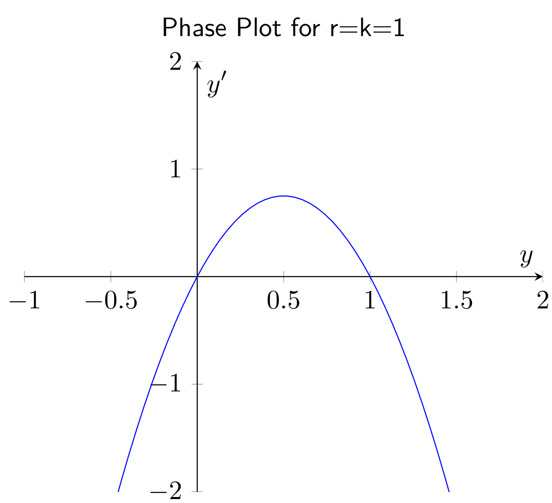
$$y = -\frac{1}{t+C}, \tag{30}$$

there is no C such that $y(0) = y_0 = 0$.

4 Multiplying like Bunnies

- Exponential growth is as follows:
$$y' = ky \implies y = Ce^{kt} \tag{31}$$
- **Logistic Growth** If uninhibited, we assume exponential growth. However, in the long run, population is limited to k. We generally have
$$y' = rh(y)y \tag{32}$$
where $h(y)$ is a limiting factor. If $h(y) = 1$, we have exponential growth, if $h(y) = 0$ we have no growth.
- We want $h(y) \approx 1$ if y is small:
  - $0 < h(y) < 1$ if $y < k$
  - $h(y) = 0$ if $y = k$
  - $h(y) < 0$ if $y > k$

  The simplest function that satisfies these is
$$y = 1 - \frac{y}{k} \tag{33}$$
so we have the equation
$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y, \tag{34}$$
which is an autonomous ODE. This allows us to draw a phase plot:



which is unstable at $x = 0$ and stable at $x = k$.

Example 10: Refer to the previous example. For what values of y does a solution have an inflection point? We have
$$y' = r(1 - y/k)y \tag{35}$$
and differentiating
$$y'' = \frac{d}{dt}y' = \frac{d}{dt}(r(1 - y/k)y) \tag{36}$$
$$= r\frac{d}{dt}\left(y - \frac{y^2}{k}\right) \tag{37}$$
$$= r\left(1 - \frac{2y}{k}\right)\frac{dy}{dt} \tag{38}$$
and inflection points occur at $y = 0, k, k/2$.

Warning: The definition of inflection point we will use in this course is simply $y'' = 0$, which is a weaker version of what we learned in ESC194.

5 Systems of Differential Equations

- We can write systems of differential equations in terms of matrices. For example:

$$\boldsymbol{x}' = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 300 \\ 300 \end{bmatrix} \tag{39}$$

- We can have a new definition of equilibrium

Definition: An equilibrium of a system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$ is the values $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$0 = \boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b} \tag{40}$$

Note: These are also known as **critical points** sometimes.

- This is just a linear algebra problem, so the solution is just

$$\boldsymbol{x} = -A^{-1}\boldsymbol{b}, \tag{41}$$

but this only works if A is invertible.

- **Direction Fields and Orbits:** Without solving an autonomous system, we can draw its **direction field**

$$\vec{F}(x) = A\vec{x} + b \tag{42}$$

on a grid of points. By convention, we draw all vectors at the same length:

  –

Example 11: Suppose we have the differential equation

$$\boldsymbol{x}' = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 300 \\ 300 \end{bmatrix}. \tag{43}$$

We can find the direction vector at a point $\vec{x} = \begin{bmatrix} 200 \\ 900 \end{bmatrix}$ by plugging it in to get

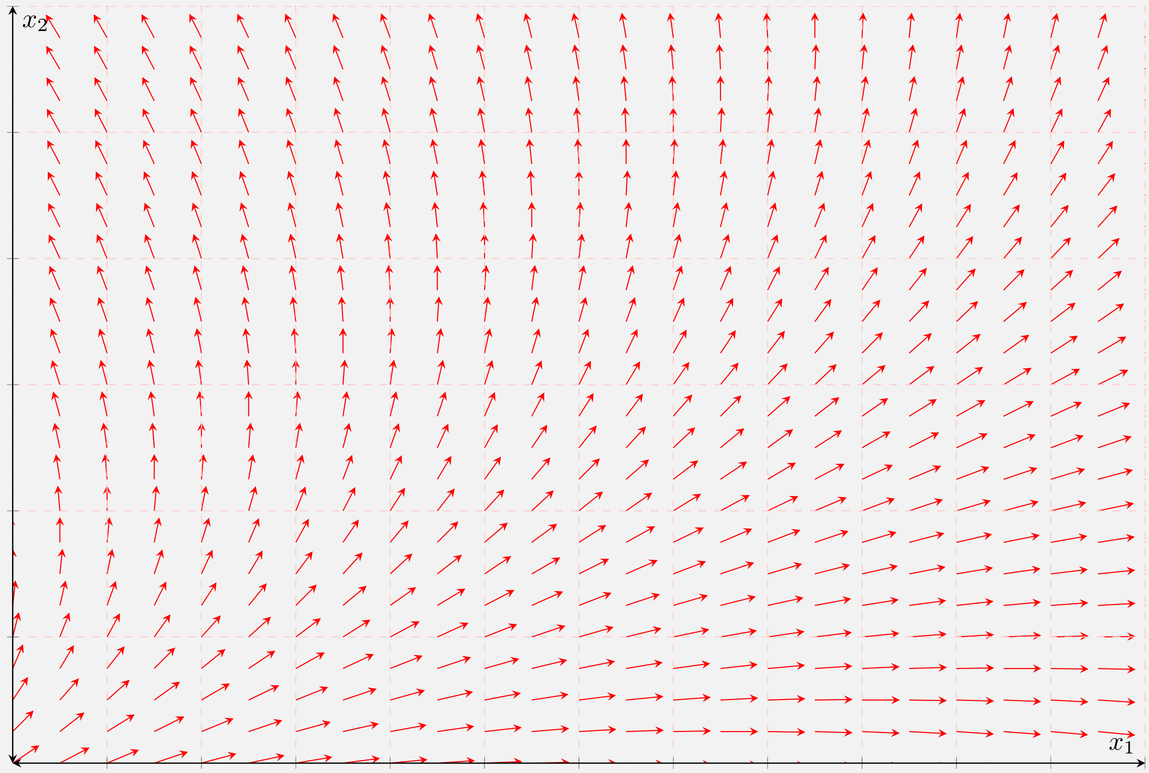
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \boldsymbol{x}' = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 200 \\ 900 \end{bmatrix} + \begin{bmatrix} 300 \\ 300 \end{bmatrix} = \begin{bmatrix} -1100 \\ 2800 \end{bmatrix}, \tag{44}$$

which after normalizing gives

$$\begin{bmatrix} -0.37 \\ 0.93 \end{bmatrix} \tag{45}$$

For example, the phase portrait looks like

Phase Portrait



From this, we can see how initial conditions can determine the end conditions.

- A **homogenous system** is when $\boldsymbol{b} = 0$.
- We can reduce a non-homogenous system to a homogenous system. Let us write

$$\boldsymbol{x} = \boldsymbol{\phi} + \boldsymbol{q}. \tag{46}$$

where $\boldsymbol{q}$ is the equilibrium. Then[1]:

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b} \iff \boldsymbol{\phi}' = A(\boldsymbol{\phi} + \boldsymbol{q}) + \boldsymbol{b} \tag{47}$$

$$\iff \boldsymbol{\phi}' = A\boldsymbol{\phi} + \underbrace{A\boldsymbol{q} + A\boldsymbol{b}}_{\text{zero since equilibrium}} \tag{48}$$

$$\iff \boldsymbol{\phi}' = A\boldsymbol{\phi} \tag{49}$$

Idea: Every solution of the non-homogenous problem can be written as a solution of the homogenous problem plus the equilibrium.

Example 12: Consider the system $\boldsymbol{x}' = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} \boldsymbol{x}$. Then we can verify that

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-2t} \tag{50}$$

solves this system by directly substitution. We can also verify that

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} \tag{51}$$

solves this equation as well. However, what is more interesting is that any linear combination of these two solutions

[1] This is similar to how in PHY293, we were able to modify the differential equation for a spring-mass system with gravity, and remove the gravity component by changing coordinates.

is also a solution: and thus it is more general.

Theorem: **Superposition Principle:** Suppose $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are solutions to $\boldsymbol{x}' = A\boldsymbol{x}$. Then for any coefficients c_1 and c_2, this is also a solution:

$$\boldsymbol{\phi}(t) = c_1\boldsymbol{\phi_1}(t) + c_2\boldsymbol{\phi_2}(t) \tag{52}$$

Alternatively, the set of solutions to $\boldsymbol{x}' = A\boldsymbol{x}$ is a subspace of functions.

- It turns out that the subspace of solutions to $\boldsymbol{x}' = A\boldsymbol{X}$ is always two-dimensional if it is a system of two equations.

Definition: Two functions $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are linearly independent if and only if

$$c_1\boldsymbol{\phi_1} + c_2\boldsymbol{\phi_2} = 0 \implies c_1 = c_2 = 0. \tag{53}$$

Example 13: We can show that $\boldsymbol{\phi_1} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$ and $\boldsymbol{\phi_2} = \begin{bmatrix} \sin t \\ e^{-t} \end{bmatrix}$ are linearly independent because

$$c_1\boldsymbol{\phi_1} + c_2\boldsymbol{\phi_2} = 0 \implies \begin{cases} c_1 e^t + c_2 \sin t = 0 \\ c_1 e^{-t} + c_2 e^{-t} = 0 \end{cases} \tag{54}$$

This linear system gives $c_1 = c_2 = 0$.

Theorem: The solutions $\boldsymbol{\phi_1}(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}$ and $\boldsymbol{\phi_2}(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$ of $\boldsymbol{x}' = A\boldsymbol{x}$ are linearly independent if and only if the Wronskian is nonzero:

$$W[\boldsymbol{\phi_1}, \boldsymbol{\phi_2}] = \det \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \neq 0 \tag{55}$$

for all $t \in I$.

Example 14: Let's solve $\boldsymbol{\phi}' = A\boldsymbol{\phi}$ where $A = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & \\ -3/4 & \end{bmatrix}$. We can use an educated guess: $\boldsymbol{\phi} = e^{\lambda t}\boldsymbol{v}$ for some vector $\boldsymbol{v}$. If we substitute this in, we get

$$A\boldsymbol{\phi} = \lambda\boldsymbol{\phi} \tag{56}$$

We can solve for the eigenvalue and eigenvectors to get

$$\boldsymbol{\phi_1} = e^{-(1/4)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{57}$$

$$\boldsymbol{\phi_2} = e^{-t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \tag{58}$$

Finally, we need to check the Wronskian is nonzero (it is). The general solution is thus:

$$x(t) = \begin{bmatrix} 1200 \\ 1200 \end{bmatrix} + c_1 e^{-(1/4)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \tag{59}$$

- We can look at some casework where we have two distinct eigenvalues:
  - $\lambda_1, \lambda_2 < 0$
    - Critical Point(s): $(0, 0)$ only.
    - Stable equilibrium
  - $\lambda_1, \lambda_2 > 0$

    * Critical Point(s): $(0, 0)$ only.
    * Unstable Equilibrium
  - $\lambda_1 > 0 > \lambda_2$
    *
  - $\lambda_1 = 0$, $\lambda_2 < 0$
    * Equilibrium: The line associated with $\lambda_1 = 0$.

Note that the $\lambda_1 = 0, \lambda_2 > 0$ case is not covered, but it is similar to the fourth case.