MAT292
Tutorial 2 Solution

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1. (a) $(x, y) = (0, vt)$

   (b) $(x, y) = (0, -b + ut)$

   Alternatively, we can write $y'_{\text{lion}}(t) = u$. Integrating and using the initial position gives the same result as above.

2. (a) It will be a concave up curve, $dx/dt > 0$.

   (b) It will pass the vertical line test.

   (c) $u = \sqrt{x'^2 + y'^2}$

   (d) We have $\dfrac{dy}{dx} = \dfrac{vt - y}{0 - x}$

3. (a) Using the chain rule, we have

$$\frac{d}{dt} y(x(t)) = \frac{dy}{dx}\frac{dx}{dt} \tag{1}$$

   Using this, we have

$$u^2 = \left(\frac{dy}{dx}\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 \tag{2}$$

   and isolating for $\dfrac{dx}{dt}$ gives

$$\boxed{\frac{dx}{dt} = \frac{u}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}} \tag{3}$$

   (b) Taking $\dfrac{dy}{dx} = \dfrac{y - vt}{x}$ and differentiating, we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{x}\left(\frac{dy}{dx} - v\frac{dt}{dx}\right) + \frac{vt - y}{x^2} && (4) \\ &= \frac{1}{x}\left(y' - v\frac{dt}{dx}\right) - \frac{y'}{x} && (5) \\ &= -\frac{v}{x}\frac{dt}{dx} && (6) \end{aligned}$$

   which gives

$$\boxed{\frac{dx}{dt} = -\frac{\frac{v}{x}}{\frac{d^2y}{dx^2}}} \tag{7}$$

   (c) See the boxed equations. Equating these, we have

$$\frac{u}{\sqrt{1 + y'^2}} = -\frac{\frac{v}{x}}{\frac{d^2y}{dx^2}} \tag{8}$$

We can make the substitution $w = y' = \dfrac{dy}{dx}$. Then the equation becomes

$$\frac{dw}{dx} = \frac{-v}{ux}\sqrt{1+w^2}. \tag{9}$$

Separating variables, we get

$$\int \frac{1}{\sqrt{1+w^2}}\,\mathrm{d}w = -\frac{v}{u}\int \frac{1}{x}\,\mathrm{d}x \tag{10}$$

$$\sinh^{-1}(w) = -\frac{v}{u}\ln|x| + C \tag{11}$$

We have $w = 0$ when $x = -a$, so

$$C = \frac{v}{u}\ln(a) \tag{12}$$

and so

$$w = \sinh\left(\frac{v}{u}\ln\left(-\frac{a}{x}\right)\right) \tag{13}$$

Letting $\dfrac{dy}{dx}$, we get the desired differential equation:

$$\frac{dy}{dx} = \sinh\left(\frac{v}{u}\ln\left(-\frac{a}{x}\right)\right) \tag{14}$$

Bonus: Using an integral calculator, I get

$$y = -\frac{ux^{\frac{v}{u}+1}}{2\,(v+u)\,(-1)^{\frac{v}{u}}\,a^{\frac{v}{u}}} - \frac{u\,(-1)^{\frac{v}{u}}\,a^{\frac{v}{u}}x^{1-\frac{v}{u}}}{2\,(v-u)} + C \tag{15}$$

At $x = -a$, we have $y = 0$, so we can solve for the constant of integration. To avoid writing fractions, let $f = v/u$.

$$0 = -\frac{u(-a)^{f+1}}{2(v+u)(-1)^f a^f} - \frac{u(-1)^f a^f(-a)^{1-f}}{2(v-u)} + C \tag{16}$$

$$0 = -\frac{u(-a)}{2(v+u)} - \frac{u(-a)}{2(v-u)} + C \tag{17}$$

$$C = -\frac{ua}{2}\left(\frac{1}{v+u} + \frac{1}{v-u}\right) \tag{18}$$

$$C = -\frac{ua}{2}\left(\frac{2v}{v^2-u^2}\right) \tag{19}$$

$$C = \frac{uv}{u^2-v^2}a \tag{20}$$

This gives

$$\boxed{y(x) = \frac{u(-x)}{2(v+u)}\left(-\frac{x}{a}\right)^{v/u} + \frac{u(-x)}{2(v-u)}\left(-\frac{a}{x}\right)^{v/u} + \frac{uv}{u^2-v^2}a} \tag{21}$$

where we skipped some steps factoring.