PHY180: Classical Mechanics
2017 Exam Solutions

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Problem 1

(a) We integrate with respect to time to get velocity:

$$v(t) = \int 5 - t\,\mathrm{d}t = 5t - \frac{1}{2}t^2 - 8 \tag{1}$$

where I have added in the condition $v(0) = -8\mathrm{m\,s^{-1}}$. This is zero when:

$$t^2 - 10t + 16 = 0 \implies (t-8)(t-2) = 0 \tag{2}$$

So the particle will stop at $t = 2$s and $t = 8$s.

(b) We integrate again:

$$x(t) = \int (5t - \frac{1}{2}t^2 - 8)\,\mathrm{d}t = \frac{5}{2}t^2 - \frac{1}{6}t^3 - 8t + 10 \tag{3}$$

using the condition that $x(0) = 10$. At $t = 3$s, the position of the particle is:

$$x(3) = 4\mathrm{m} \tag{4}$$

(c) A critical point for the velocity of the particle occurs when $v'(t) = a(t) = 0$, or when $t = 5$s, and the particle is travelling at a speed of:

$$v(4) = 4\mathrm{m\,s^{-1}} \tag{5}$$

Note that the initial and end points do not satisfy this as they are both moving in the $-\hat{\imath}$ direction.

Problem 2

(a) Before it hits object m, it is moving at a speed of $v = \sqrt{2gh}$, from conservation of energy.

(b) We have a perfectly inelastic collision, where momentum is conserved:

$$M\sqrt{2gh} = (M+m)v_A \implies v_A = \frac{M}{M+m}\sqrt{2gh} \tag{6}$$

(c) For the minimum speed, they are just about to leave contact with the track so the normal force becomes zero at the top of the track. Therefore:

$$(m+M)g = \frac{(m+M)v_A'^2}{R} \tag{7}$$

where from conservation of energy, v_A' is the speed at the top of the track and is given by:

$$\frac{1}{2}(M+m)v_A^2 = \frac{1}{2}(M+m)v_A'^2 + (m+M)g(2R) \implies v_A' = \sqrt{v_A^2 - 4gR} \tag{8}$$

Substituting this into the force balance equation gives:

$$gR = v_A^2 - 4gR \implies v_A = \sqrt{5gR} \tag{9}$$

Problem 3

(a) The distance the contact point between the applied force and the rod moves is given by:

$$y \tan(\Delta\theta) \tag{10}$$

so the work done is:

$$W = Fy\tan(\Delta\theta) = 3.5\text{J} \tag{11}$$

(b) Let $y = 0$ to be at the top of the rod. From conservation of energy, we have:

$$-Mg\frac{L}{2} + W_{\text{ext}} = -Mg\frac{L}{2}\cos\theta + K \tag{12}$$

Solving for K gives:

$$K = \frac{MgL}{2}\left(\cos\Delta\theta - 1\right) + W_{\text{ext}} = 1.4\text{J} \tag{13}$$

Problem 4

Consider a force balance on the mass:

$$m_2(a - \alpha R) = T - m_2 g \tag{14}$$

Note that the a comes from the acceleration of the elevator and the $-\alpha R$ comes from the acceleration of the block with respect to the spool. A torque balance gives:

$$I_{\text{cm}}\alpha = TR = m_2(g+a)R - m_2\alpha R^2 \tag{15}$$

Solving for α then gives:

$$\alpha = \frac{m_2 R(a+g)}{I_{\text{cm}} + m_2 R^2} \tag{16}$$

Problem 5

Some orbital mechanics is needed for this problem, so feel free to skip this.

(a) We can calculate the center of mass in the radial direction away from mass M to be:

$$x_{\text{cm}} = \frac{2MD}{M + 2M} = \frac{2}{3}D \tag{17}$$

Therefore, a force balance on mass M gives:

$$M\omega^2\left(\frac{2}{3}D\right) = \frac{GM(2M)}{D^2} \implies \omega = \sqrt{\frac{3GM}{D^3}} \tag{18}$$

(b) The angular velocity of both masses are the same about their centroid. Their respective velocities are $v = \frac{2}{3}\omega D$ and $v = \frac{1}{3}\omega D$, so their total kinetic energy is:

$$K = \frac{1}{2}(M)\frac{4D}{9}\frac{3GM}{D^3} + \frac{1}{2}(2M)\frac{D}{9}\frac{3GM}{D^3} = \frac{GM^2}{D^2} \tag{19}$$

(c) The potential energy is:

$$U = -\frac{GM(2M)}{D^2} = -\frac{2GM^2}{D^2} \tag{20}$$

Note that $2K + U = 0$. This is a special case of a well known result called the **Virial Theroem**.

Problem 6

Consider a tiny displacement θ of the rod. The spring would exert a torque of:

$$\tau_s = -k\Delta x \frac{L\cos\theta}{2} \tag{21}$$

Approximating $\cos\theta \approx 1$ and $\Delta x = \frac{L}{2}\theta$, we get the torque from the spring to be:

$$\tau_s = -\frac{kL^2}{4}\theta \tag{22}$$

and the torque from gravity is:

$$\tau_g = -mg\frac{L}{2}\sin\theta \approx -\frac{mgL}{2}\theta \tag{23}$$

such that the torque balance equation becomes:

$$\frac{1}{12}mL^2\alpha = -\left(\frac{mgL}{2} + \frac{kL^2}{4}\right)\theta \implies \alpha = -\left(\frac{6g}{L} + \frac{3k}{m}\right)\theta \tag{24}$$

so the angular frequency is:

$$\omega = \sqrt{\frac{6g}{L} + \frac{3k}{m}} \tag{25}$$