PHY293: Tutorial Problems
Tutorial 1 Solutions

QiLin Xue

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1. (a) The angular frequency is

$$\boxed{\omega = \sqrt{\frac{k}{m}} = 5 \text{ rad/s}}$$

(b) The total mechanical energy of the system is equal to the potential energy at maximum compression:

$$\boxed{E = \frac{1}{2}kA^2 = 0.36 \text{ J}}.$$

(c) Using conservation of energy, we have

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx(0)^2.$$

We know everything except v, so solving for it gives $v = 0.566$ m/s. We want velocity so the answer is

$$\boxed{\vec{v} = 0.566 \text{ m/s [right]}}.$$

(d) We have $x(t) = A\sin(\omega t + \phi)$. Using the initial condition $x(0) = -0.04$ m, we have $-0.333 = \sin(\phi)$. We get two solutions: $\phi = -19.47^\circ$ and $\phi = 199.5^\circ$. We want $v(0) = A\omega\cos(\phi) > 0$, so the right solution is $\boxed{\phi = 19.47^\circ}$.

(e) The spring will reach equilibrium when $\sin(\omega t + \phi) = 0$, or when $\omega t + \phi = 0$. This occurs after a time

$$\Delta t = \frac{-\phi}{\omega} = 0.0680 \text{ s}.$$

Since the spring is symmetric across equilibrium, it takes another Δt seconds to reach $+0.040$ m, so the time at which it first reaches here is

$$\boxed{t_1 = 2\Delta t = 0.136 \text{ s.}}$$

The period is $T = \frac{2\pi}{\omega} = 1.257$ s, so the time at which the block reaches this location a second time (i.e. on the way back) is:

$$\boxed{t_2 = T/2 = 0.628 \text{ s}}.$$

2. The velocity function is given by $v(t) = A\omega\cos(\omega t + \phi)$. The maximum speed is $A\omega = 2$ cm/s, and $v(0) = -1$, so we have $-1 = 2\cos(\phi)$. Solving for ϕ gives $\phi = 120^\circ$ and $\phi = 240^\circ$.

The mass is travelling to the left and is about to reach maximum elongation, so the position $x(t) = A\sin(\phi)$ is negative. Therefore, we have $\boxed{\phi = 240^\circ}$.

3. **System 1:** The angular frequency is $\omega = \sqrt{\frac{k_a}{m}}$.

   System 2: Suppose the block is moved a distance x from equilibrium. Then the equation of motion (from $F = ma$) is:

   $$ma = -k_b x + (-k_b x) = -2k_b x.$$

   Therefore, the effective spring constant is $2k_b$ and the angular frequency is $\omega = \sqrt{\frac{2k_b}{m}}$.

   System 3: We claim that the effective spring constant is $\frac{k_c}{2}$. There are three ways to see this:

   - *Method 1:* The springs are identical, so if one spring stretches by $\Delta x/2$, then the other spring will also stretch by $\Delta x/2$, resulting in a net stretch of Δx. Therefore, when the mass moves a distance x, it only experiences a force of $-k(x/2)$.
   - *Method 2:* We can view this as one long spring. Recall from CIV102 that the spring constant is $k = \frac{EA}{L}$, where E is the Young's Modulus, A is the cross sectional area, and L is the length. We're not changing E or A, but by doubling L, we are halving k.
   - *Method 3:* Let the mass move a distance d and let the spring attached to it get stretched a distance Δx_1, and let the spring attached to the wall get stretched a distance Δx_2. We want to relate Δx_1 and Δx_2. From Newton's third law, we have

     $$k_c \Delta x_1 = k_c \Delta x_2 \implies \Delta x_1 = \Delta x_2$$

     and we can continue with the same argument as method 1. However, this is a bit more general and can be used when the spring constants are not equal.

   The angular frequency is then $\omega = \sqrt{\frac{k_c}{2m}}$. These frequencies are all equal, so we have

   $$\sqrt{\frac{k_a}{m}} = \sqrt{\frac{2k_b}{m}} = \sqrt{\frac{k_c}{2m}} \implies k_a = 2k_b = k_c/2$$

   which leads to the desired ratio of

   $$\boxed{k_a : k_b : k_c = 1 : 1/2 : 2.}$$

4. We can consider the forces acting on it. There is a gravitational force mg downwards and a normal force N upwards. The mass will loss contact with the platform when $N = 0$, i.e. when the acceleration of the block is g.

   The platform moves with a position $y(t) = A\sin(\omega t)$ where $\omega = 2\pi f = 15.7$ rad/s. Then the acceleration is $a(t) = A\omega^2 \sin(\omega t)$ and reaches a maximum value of $A\omega^2$. This is equal to g when

   $$\boxed{A = g/\omega^2 = 0.04 \text{ m}}.$$

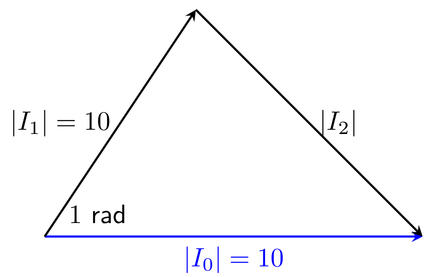
5. (a) We want to find an amplitude A such that

   $$10\cos(\omega t) = 10\cos(\omega t + 1) + A\cos(\omega t + \phi), \tag{1}$$

   or alternatively:

   $$\begin{aligned} 10\cos(\omega t) - 10\cos(\omega t + 1) &= 10\left(-2\sin\left((\omega t + \omega t + 1)/2\right)\cos\left((\omega t - \omega t - 1)/2\right)\right) \\ &= -20\sin(\omega t + 0.5)\sin(-1/2) \\ &= 9.59\sin(\omega t + 0.5) \end{aligned}$$

   where we used the difference in cosine formula. Therefore, the magnitude is $\boxed{|I_2| = 9.59 \text{ A}}$. There is actually a nice geometric interpretation of this using phasors. We want:



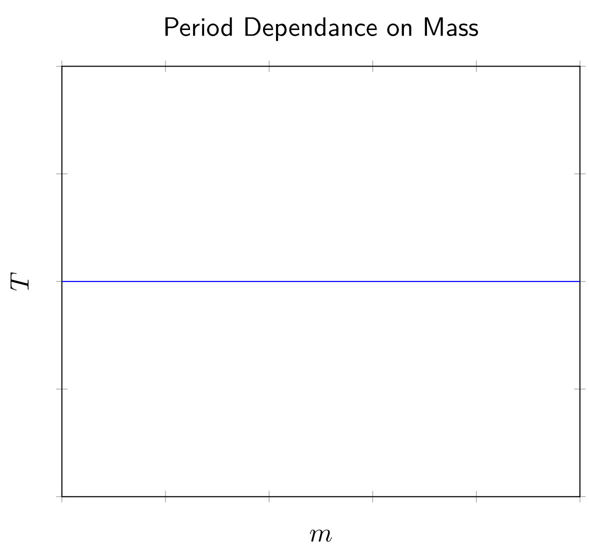
The sum of the two phasors $I_1 + I_2$ needs to be horizontal (which represents no phase shift), and we can determine this using cosine law:

$$|I_2| = \sqrt{10^2 + 10^2 - 2(10)(10)\cos(1)} = 9.59. \tag{2}$$

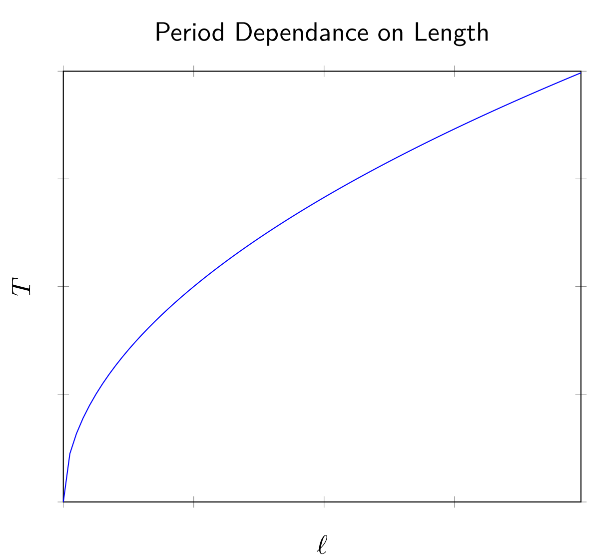
(b) Using the same idea, note that the smallest $|I_2|$ such that the sum of the two phasors lie on the horizontal line is if I_2 is perpendicular to I_0, in which we can use standard trigonometry:

$$\boxed{I_2 = I_1 \sin(1) = 8.41 \text{ A}}. \tag{3}$$

6. The period does not depend on the mass, so we have



The period depends on length as $T \propto \sqrt{\ell}$, so we have:



7. (a) Using trigonometry, we have

$$\boxed{\theta = \sin^{-1}\left(\frac{0.0350}{0.750}\right) = 2.67^\circ} \tag{4}$$

The small angle approximation can be used.

Note: To be more complete, using this θ results in approximately a 0.04% error. This is much less error than possible uncertainties in the length and position measurements.

(b) The initial height (measured from the top) is

$$h = \sqrt{0.750^2 - 0.0350^2} = 0.74918 \text{ m}. \tag{5}$$

so there is a height difference of $\Delta h = 0.000817$ m between the maximum and lowest point. Energy conservation tells us the maximum speed is

$$\boxed{v = \sqrt{2g\Delta h} = 0.127 \text{ m/s}}. \tag{6}$$

(c) This is a fourth of the period, so the time to reach this speed is

$$\boxed{t = T/4 = \frac{1}{4} \cdot 2\pi\sqrt{\frac{\ell}{g}} = 0.434 \text{ s.}} \tag{7}$$

8. (a) The relationship between a conservative force and potential energy is

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{8}$$

Applying this, we get

$$\boxed{F = \frac{6a}{x^7} - \frac{12b}{x^{13}}}. \tag{9}$$

(b) Equilibrium occurs when $F = 0$, or when:

$$\frac{6a}{x^7} = \frac{12b}{x^{13}} \implies \boxed{x_0 = \left(\frac{2b}{a}\right)^{1/6}}. \tag{10}$$

(c) The angular frequency at equilibrium is equal to

$$\omega = \sqrt{\frac{U''(x)}{m}} = \sqrt{\frac{F'(x)}{m}}. \tag{11}$$

Taking the derivative again, we get

$$F'(x) = -\frac{6a}{x^7}\frac{7}{x} + \frac{12b}{x^{13}}\frac{13}{x}. \tag{12}$$

Letting $\frac{6a}{x^7} = \frac{12b}{x^{13}}$, we have:

$$\begin{aligned} F'(x_0) &= -\frac{6a}{x_0^7}\frac{7}{x_0} + \frac{6a}{x_0^7}\frac{13}{x_0} \\ &= \frac{6a}{x_0^6}\left(\frac{13}{x_0^2} - \frac{7}{x_0^2}\right) \\ &= \frac{6a}{x_0^6}\left(\frac{6}{x_0^2}\right) \end{aligned}$$

Letting $x_0 = \left(\frac{2b}{a}\right)^{1/6}$ and substituting this in, we have

$$\boxed{F'(x_0) = 36a\frac{a}{2b}\left(\frac{a}{2b}\right)^{1/3} = 36a\left(\frac{a}{2b}\right)^{4/3}.} \tag{13}$$