Random Walks in a Vibrating Chain: A More Generalized Approach to Studying Unknotting

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(Dated: October 6, 2022)

In this experiment, a trefoil knot was tied in the center of chains made up of 3.9 mm diameter brass beads$^{(\text{pg } 9)}$. When vibrated on a flat aluminum plate at a frequency of 17 Hz and accelerations of $2.6g - 3.5g$, the three crossings on the trefoil knot exhibit stochastic behavior, and given sufficient time, will unknot. Various lengths were tested, ranging from $N = 36$ to $N = 80$ beads$^{(\text{pg } 6-7,9)}$. The apparatus setup followed Professor David Bailey's instructions, but the mathematical model we worked with was based off of E. Ben-Naim et al's work[1]. Ben-Naim's experiment showed that the average unknotting time scaled like

$$\tau \sim (N - N_0)^2,$$

where N_0 is the number of beads involved in the knot. This is able to be derived by modelling the motion of the crossing points as a random walk. Our experiment will refute this claim and show that under certain circumstances, the unknotting time scales linearly$^{(\text{pg } 21-27)}$. The reason for this difference is that under certain chain geometries and physical conditions, the crossing points may display an asymmetric walk and there is a preference for the three crossing points to move in a certain direction, namely$^{(\text{pg } 10-13)}$:

$$\tau \sim \frac{2N - N_0}{4p - 2} \cdot \frac{\kappa^{N_0/2} - \kappa^{N/2}}{\kappa^{N_0/2} + \kappa^{N/2}},$$

where $\kappa = \frac{1}{p} - 1$. In the $p = 0.5$ case, the limit approaches $\frac{1}{4}(N - N_0)^2$ as predicted. In fact, for any $p \neq 1/2$, our model and computer simulation$^{(\text{pg } 32)}$ shows that the asymptotic behavior is linear.

Further proof of this asymmetry is found by noting there is a particular preference for the knot to travel in a certain direction$^{(\text{pg } 8)}$. Namely, if the end of the chain enters the trefoil knot by going over the knot$^{(\text{pg } 5)}$, then the knot will always end up in that direction. This motivates the idea of tying the knot off-center at a location that is far away from the favored end$^{(\text{pg } 14-16)}$, in an attempt to counter the asymmetry. We derive that if the knot is placed rN away from the unfavored end, where $0 < r < 0.5$, then the probability it will reach the unfavored end of the chain is given by$^{(\text{pg } 16-17)}$

$$P = \left(\frac{1}{p} - 1\right)^{rN}.$$

Our experimental results$^{(\text{pg } 40)}$ show an exponential decay that is similar to the above.

Slow motion video of the stochastic nature of the knot was taken, and analyzed by frames$^{(\text{pg } 41-45)}$. The main challenge was that it was very time exhaustive. However, this analysis was able to reveal some inconsistencies of our generalized model$^{(\text{pg } 45)}$.

Because the behavior of the knot under vibrations is a stochastic process, the majority of the uncertainties come from simply not collecting enough data to make a definitive statement about averages. If we are to ensure the set-up procedure is always consistent,$^{(\text{pg } 5)}$ then each trial can take a considerable amount of time, and our data analysis has shown that 30 trials per condition is not enough to make conclusive justifications.

[1] E. Ben-Naim, Z. A. Daya, P. Vorobieff, and R. E. Ecke. Knots and random walks in vibrated granular chains. *Physical Review Letters*, 86(8):1414–1417, February 2001.