# PHY365: Quantum Information
# Problem Set 1

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1. (a) We can compute $\det\left(\hat{X} - I\lambda\right) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1$. This is zero if and only if $\lambda = \pm 1$, which are the eigenvalues. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda = 1, -1$ respectively. The diagonal representation is then

$$\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \tag{0.1}$$

   (b) Via the same process, we have

$$\hat{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{bmatrix} \tag{0.2}$$

   (c) and

$$\hat{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{0.3}$$

   **Remarks:** The three Pauli matrices form a group, known as the Pauli group $G_1 = \langle \hat{X}, \hat{Y}, \hat{Z} \rangle$.

2. These can be shown by straight matrix multiplication. However, we can use the special trick that

$$\hat{X}\hat{Y}\hat{Z} = iI \tag{0.4}$$

and $\hat{X}, \hat{Y}, \hat{Z}$ are all order 2. The first and third relations then follow immediately. The second relation also follows immediately from their anti-commutative property. Thus:

$$\hat{X}\hat{Z}\hat{X} = i\hat{X}\hat{Y} \tag{0.5}$$

$$= i^2\hat{Z}. \tag{0.6}$$

3. We can compute

$$\hat{U}^\dagger = \cos\alpha \hat{I} + \sin\alpha \left(\sin\theta\cos\phi\hat{X}^\dagger + \sin\theta\sin\phi\hat{Y}^\dagger + \cos\theta\hat{Z}^\dagger\right) \tag{0.7}$$

$$= \cos\alpha \hat{I} + \sin\alpha \left(\sin\theta\cos\phi\hat{X} - \sin\theta\sin\phi\hat{Y} + \cos\theta\hat{Z}\right) \tag{0.8}$$

4. Let us compute $\hat{U}, \hat{U}^\dagger$:

$$\hat{U} = \begin{bmatrix} \cos\alpha + i\sin\alpha\cos\theta & \sin\alpha\sin\theta\sin\phi + i\sin\alpha\sin\theta\cos\phi \\ -\sin\alpha\sin\theta\sin\phi + i\sin\alpha\sin\theta\cos\phi & \cos\alpha - i\sin\alpha\cos\theta \end{bmatrix} \tag{0.9}$$

$$\hat{U}^\dagger = \begin{bmatrix} \cos\alpha - i\sin\alpha\cos\theta & -\sin\alpha\sin\theta\sin\phi - i\sin\alpha\sin\theta\cos\phi \\ \sin\alpha\sin\theta\sin\phi - i\sin\alpha\sin\theta\cos\phi & \cos\alpha + i\sin\alpha\cos\theta \end{bmatrix} \tag{0.10}$$

and their product by looking at each element

$$
\begin{aligned}
a_{11} &= (\cos^2\alpha + \sin^2\alpha\cos^2\theta) + (\sin^2\alpha\sin^2\theta\sin^2\phi + \sin^2\alpha\sin^2\theta\cos^2\phi) \\
&= \cos^2\alpha + \sin^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta \\
&= \cos^2\alpha + \sin^2\alpha \\
&= 1 \\
a_{12} &= (\cos\alpha + i\sin\alpha\cos\theta)(-\sin\alpha\sin\theta\sin\phi - i\sin\alpha\sin\theta\cos\phi) \\
&\quad + (\sin\alpha\sin\theta\sin\phi + i\sin\alpha\sin\theta\cos\phi)(\cos\alpha + i\sin\alpha\cos\theta) \\
&= 0
\end{aligned}
$$

Due to similar patterns, we can indeed verify that $a_{21} = 0$ and $a_{22} = 0$. Therefore, $\hat{U}\hat{U}^\dagger = \hat{I}$ implies that $\hat{U}$ is unitary.