AER372: Control Systems
Assignment 2

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2.1 (a) We have the following array:

s^5	1	30	344
s^4	10	80	480
s^3	$\frac{300-80}{10}=22$	$\frac{344\cdot 10-1\cdot 480}{10}=296$	0
s^2	$\frac{80\cdot 22-10\cdot 296}{22}=-\frac{600}{11}$	480	0
s^1	$\frac{296\cdot -600/11-480\cdot 22}{-600/11}=\frac{2448}{5}$	0	
s^0	480		

There are 2 sign changes, so there are two roots with positive real parts.

(b) We have the following array:

s^4	1	7	8
s^3	2	-2	0
s^2	$\frac{2\cdot 7-1\cdot -2}{2}=8$	8	
s^1	$\frac{-2\cdot 8-2\cdot 8}{8}=-4$	0	
s^0	8		

Again, there are two sign changes, so there are two roots with positive real parts.

(c) We have the following array:

s^4	1	6	25
s^3	0	0	0
s^3 (new)	4	12	0
s^2	$\frac{24-12}{4}=3$	25	
s^1	$\frac{12\cdot 3-4\cdot 25}{3}=-\frac{64}{3}$	0	
s^0	25		

where the new row for s^3 was created by considering the auxiliary polynomial for s^4, which was s^4+6s^2+25, and computing its derivative, which was $4s^3+12s$.

2.2 (a) The transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{e^{-sT} \cdot \frac{A}{s(s+1)}}{1 + e^{-sT} \cdot \frac{A}{s(s+1)}} \tag{0.1}$$

$$= \frac{A}{A + s(s+1)e^{Ts}}, \tag{0.2}$$

so the characteristic equation is

$$A + s(s+1)e^{Ts} = 0. \tag{0.3}$$

(b) Making this substitution, the transfer function becomes:

$$\frac{Y(s)}{R(s)} = \frac{(1 - sT) \cdot \frac{A}{s(s+1)}}{1 + (1 - sT) \cdot \frac{A}{s(s+1)}} \tag{0.4}$$

$$= \frac{A(Ts - 1)}{A(Ts - 1) - s(s+1)}, \tag{0.5}$$

and we wish to solve for the stability of the characteristic polynomial,

$$s^2 + (1 - AT)s + A = 0 \tag{0.6}$$

by using the Routh's Stability criterion. The Routh array is:

s^2	1	A
s^1	$1 - AT$	0
s^0	A	0

For the system to be stable, we need no sign changes in the first column. This means that $AT < 1$ and $A > 0$. If we make a different approximation for e^{-sT}, we get the transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{1-sT/2}{1+sT/2} \cdot \frac{A}{s(s+1)}}{1 + \frac{1-sT/2}{1+sT/2} \cdot \frac{A}{s(s+1)}} \tag{0.7}$$

$$= \frac{A(Ts - 2)}{A(Ts - 2) - s(s+1)(Ts + 2)} \tag{0.8}$$

$$= \frac{A(Ts - 2)}{ATs - 2A - Ts^3 - Ts^2 - 2s^2 - 2s} \tag{0.9}$$

$$= \frac{A(2 - Ts)}{Ts^3 + (T+2)s^2 + (2 - AT)s + 2A}. \tag{0.10}$$

The Routh array for the characteristic polynomial is

s^3	T	$2 - AT$
s^2	$T + 2$	$2A$
s^1	$-\dfrac{2AT - (2 - AT)(2 + T)}{T + 2}$	0
s^0	$2A$	0.

We want the first column to all have the same sign. Note that since $T > 0$ we have $T + 2 > 0$. We just need $A > 0$ and

$$2AT - (2 - AT)(2 + T) < 0 \implies AT^2 + 4AT - 2T - 4 < 0. \tag{0.11}$$

2.3 (a) First, compute the transfer function for system (a). It is consisted of two nested negative feedback loops. The inner loop has a gain of

$$\frac{\frac{KK_m}{s(1+\tau_m s)}}{1 + k_t s \frac{KK_m}{s(1+\tau_m s)}} = \frac{KK_m}{s(KK_m k_t + s\tau_m + 1)}. \tag{0.12}$$

Therefore, the transfer function is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{\frac{k_P K K_m/k}{s(KK_m k_t + s\tau_m + 1)}}{1 + \frac{k_P K K_m/k}{s(KK_m k_t + s\tau_m + 1)}} \tag{0.13}$$

$$= \frac{KK_m k_P/k}{KK_m k_P/k + s(KK_m k_t + s\tau_m + 1)}. \tag{0.14}$$

The transfer function for (b) is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{\frac{K'}{s(1+\tau_m s)}}{1 + (1 + k_t' s)\frac{K'}{s(1+\tau_m s)}} \tag{0.15}$$

$$= \frac{K'}{K' + s(K' k_t' + s\tau_m + 1)}. \tag{0.16}$$

Matching coefficients, we obtain

$$K' = \frac{KK_m k_P}{k} \tag{0.17}$$

$$k_t' = \frac{k_t k}{k_P}. \tag{0.18}$$

(b) This is a unity feedback control, so the open-loop transfer function is

$$GD_{\text{cl}} = \frac{k_P K K_m/k}{s(KK_m k_t + s\tau_m + 1)}, \tag{0.19}$$

which has a single pole at $s = 0$, so according to the Theorem learned in class, it is type 1, and the velocity error coefficient is

$$K_\nu = \frac{k_P K K_m/k}{(KK_m k_t + 1)} = \frac{K'}{1 + k_t'}. \tag{0.20}$$

(c) Since k_t is directly proportional to k_t' and not K', increasing k_t will cause the denominator to grow, which decreases K_ν.

2.4 (a) We obtain

$$E_c(s) = R(s) - Y_c(s) = R\left(1 - \frac{D_c G}{1 + D_c GH}\right) = R(s) \cdot \frac{(D_c GH - D_c G + 1)}{D_c GH + 1}, \tag{0.21}$$

so the transfer function is

$$\frac{E_c(s)}{R(s)} = \frac{D_c(s)G(s)H(s) - D_c(s)G(s) + 1}{D_c(s)G(s)H(s) + 1}. \tag{0.22}$$

A ramp reference input is given by $r(t) = t1(t) \implies R(s) = \frac{1}{s^2}$. Therefore,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{D_c(s)G(s)H(s) - D_c(s)G(s) + 1}{s(D_c(s)G(s)H(s) + 1)} \tag{0.23}$$

(b) Write $G(s) = \frac{\tilde{G}(s)}{s}$. Then to be Type 1, it needs to be able to track a ramp reference input, i.e. $|e_{ss}| < \infty$. We obtain,

$$e_{ss} = \lim_{s\to 0} \frac{\frac{1}{s}0.73\tilde{G}(s)(H(s) - 1) + 1}{s(\frac{1}{s}0.73\tilde{G}(s)H(s) + 1)} \tag{0.24}$$

$$= \lim_{s\to 0} \frac{100s}{s(100s + 73H(s)\tilde{G}(s))} + \lim_{s\to 0} \frac{73(H(s) - 1)\tilde{G}(s)}{s(100s + 73H(s)\tilde{G}(s))} \tag{0.25}$$

$$= \frac{100}{73H\tilde{G}} + \lim_{s\to 0} \frac{73(H(s) - 1)\tilde{G}(s)}{s(100s + 73H(s)\tilde{G}(s))}. \tag{0.26}$$

For the second term to not diverge, we want to write

$$(H(s) - 1)\tilde{G}(s) = sA(s) \tag{0.27}$$

for some $A(s)$ which does not have a pole at $s = 0$. That is, we want $H(s)$ in the form of

$$H(s) = \frac{sA(s)}{\tilde{G}(s)} + 1 = \frac{A(s)}{G(s)} + 1. \tag{0.28}$$

If this was true, then we have:

$$e_{ss} = \frac{100}{73\tilde{G}(0)} + \frac{A(0)}{\tilde{G}(0)} = \frac{A(0) + \frac{100}{73}}{\tilde{G}(0)}. \tag{0.29}$$

We just need to be careful that $A(s)$ does not have a pole at $s = 0$. Note that we satisfy the relationship $H(0) = 1$ since $\tilde{G}(0) \neq 0$.

(c) Plugging this in, we have

$$\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{s}{s^2} \frac{1 + (H-1)GD}{1 + DGH} && (0.30)\\
&= \lim_{s\to 0} \frac{1}{s} \frac{1 + \left(\frac{2.75s+1}{0.36s+1} - 1\right)\frac{1}{s(s+1)^2} \cdot 0.73}{1 + \frac{1}{s(s+1)^2} \cdot 0.73 \cdot \frac{2.75s+1}{0.36s+1}} && (0.31)\\
&= \lim_{s\to 0} \frac{400(s+1)^2 \cdot (9s+25) + 17447}{25 \cdot (16s(s+1)^2 \cdot (9s+25) + 803s + 292)} && (0.32)\\
&= \frac{400(0+1)^2 \cdot (0+25) + 17447}{25 \cdot (0(0+1)^2 \cdot (0+25) + 0 + 292)} && (0.33)\\
&= 3.75986301369863. && (0.34)
\end{aligned}$$

Thus, $K_\nu = 3.75986301369863^{-1} = 0.26597\ \text{s}^{-1}$.

2.5 (a) Because it is unity feedback, the open loop transfer function is

$$D_c(s)G(s) = \frac{K(s+a)}{s+b} \cdot \frac{1}{s^2 + 2\zeta s + 1} = \frac{K(a+s)}{(b+s)(s^2 + 2s\zeta + 1)}. \tag{0.35}$$

To be type 1, we need $b = 0$ to get $s = 0$ as a pole. However, we also need $a, K \neq 0$ in order to prevent cancelling out the pole.

(b) The closed loop transfer function is

$$\begin{aligned}
T(s) = \frac{D_c(s)G(s)}{1 + D_c(s)G(s)} &= \frac{\frac{K(a+s)}{(0+s)(s^2+2s\zeta+1)}}{1 + \frac{K(a+s)}{(0+s)(s^2+2s\zeta+1)}} && (0.36)\\
&= \frac{K(a+s)}{K(a+s) + s(s^2 + 2s\zeta + 1)} && (0.37)\\
&= \frac{K(a+s)}{s^3 + 2\zeta s^2 + (K+1)s + aK}. && (0.38)
\end{aligned}$$

We can create a Routh array for this transfer function to determine the stability,

s^3	1	$K+1$
s^2	2ζ	aK
s^1	$\frac{(K+1)2\zeta - aK}{2\zeta}$	
s^0	aK	

To ensure that the system is stable, we require the first column to all be positive. That is, $\zeta > 0$, $aK > 0$, and

$$2\zeta(K+1) > aK \implies K > \frac{2\zeta}{a - 2\zeta}. \tag{0.39}$$

(c) We can solve for a to get

$$\frac{2\zeta(K+1)}{K} > a \tag{0.40}$$

For this to be true for all values of a, we can find the range of the LHS as a function of K in the domain $(0, \infty)$. We obtain the range $(2\zeta, \infty)$. Therefore,

$$0 < a < 2\zeta, \qquad b = 0, \qquad \zeta > 0, \tag{0.41}$$

if and only if the system is both Type 1 and remains stable for every positive value for K.

2.6 (a) Because it has unity feedback, the system type is equal to the poles of

$$D_c(s)G(s) = \frac{10(s+2)}{s^2(s+5)}, \tag{0.42}$$

so it is 2. We can compute

$$K_a = K_2 = \lim_{s\to 0} \frac{10(s+2)}{s+5} = \frac{10(2)}{5} = 4. \tag{0.43}$$

Therefore, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{4}$ and $e_{ss} = 0$ for lower order inputs.

(b) We have shown in lecture that the transfer function when taking into account the disturbance W for a unity feedback system is

$$-T_w(s) = \frac{E_c(s)}{W(s)} = \frac{-G(s)}{1 + G(s)D_c(s)} \tag{0.44}$$

$$= \frac{-1/s^2}{1 + \frac{10(s+2)}{s^2(s+5)}} \tag{0.45}$$

$$= \frac{-s-5}{s^3 + 5s^2 + 10s + 20}, \tag{0.46}$$

which has no zeros at the origin, so it is of type 0. The error is then

$$e_{ss} = -T_w(0) = -\frac{5}{20} = -\frac{1}{4}, \tag{0.47}$$

and $K_{0,w} = 4$.

2.7 Let us define

$$D_c(s) = 160 \cdot \frac{s+4}{s+30} \tag{0.48}$$

and

$$G(s) = \frac{1}{s(s+2)}. \tag{0.49}$$

Note that we have unity feedback, so the standard formulas from lecture apply.

(a) We have

$$D_c(s)G(s) = 160 \cdot \frac{s+4}{s+30} \cdot \frac{1}{s(s+2)} = \frac{160(s+4)}{s(s+2)(s+30)}, \tag{0.50}$$

which is type 1, so it can track a step reference input with zero steady-state error. The velocity constant is

$$K_\nu = 160 \cdot \frac{4}{30} \cdot \frac{1}{2} = \frac{32}{3}. \tag{0.51}$$

(b) We can compute

$$T_w(s) = \frac{\frac{1}{s(s+2)}}{1 + \frac{1}{s(s+2)} \cdot 160 \cdot \frac{s+4}{s+30}} \tag{0.52}$$

$$= \frac{s+30}{s^3 + 32s^2 + 220s + 640}, \tag{0.53}$$

which has no zeros at the origin, so it is of type 0, so it cannot reject a step disturbance w with zero steady-state error.

(c) The sensitivity for unity feedback control is

$$\mathcal{S}_G^T = \frac{1}{1+GD_c} = \frac{1}{1+\frac{1}{s(s+2)}\cdot 160\cdot\frac{s+4}{s+30}} = \frac{s(s+2)(s+30)}{s^3+32s^2+220s+640}. \tag{0.54}$$

Let $p=2$ and write the gain as

$$G(s) = \frac{1}{s(s+p)}. \tag{0.55}$$

We have,

$$\left.\mathcal{S}_p^G\right|_{p=2} = \frac{p}{\frac{1}{s(s+p)}}\frac{\partial}{\partial p}\left(\frac{1}{s(s+p)}\right)\bigg|_{p=2} \tag{0.56}$$

$$= -\frac{p}{p+s}\bigg|_{p=2} \tag{0.57}$$

$$= -\frac{2}{s+2}. \tag{0.58}$$

Therefore,

$$\mathcal{S}_2^T = \mathcal{S}_G^T\mathcal{S}_2^G \tag{0.59}$$

$$= \frac{1}{1+\frac{1}{s(s+2)}\cdot 160\cdot\frac{s+4}{s+30}}\cdot -\frac{2}{s+2} \tag{0.60}$$

$$= -\frac{2s(s+30)}{s(s+2)(s+30)+160s+640}. \tag{0.61}$$

As $s\to 0$, the sensitivity approaches 0.

(d) For $H(s) = \dfrac{20}{s+20}$, we have the transfer function

$$\frac{E_c(s)}{R(s)} = 1 - \frac{D_c(s)G(s)}{1+D_c(s)G(s)H(s)} \tag{0.62}$$

$$= 1 - \frac{\frac{160(s+4)}{s(s+2)(s+30)}}{1+\frac{20}{s+20}\cdot\frac{160(s+4)}{s(s+2)(s+30)}} \tag{0.63}$$

$$= \frac{s(s^3+52s^2+540s+560)}{s^4+52s^3+700s^2+4400s+12800} \tag{0.64}$$

For a unit-step, we have $R(s) = \dfrac{1}{s}$, so the error is

$$e_{ss} = \lim_{s\to 0} E_c(s) = 0, \tag{0.65}$$

so yes, it can track a step reference input with zero steady-state error. We can compute

$$T_w(s) = \frac{\frac{1}{s(s+2)}}{1+\frac{20}{s+20}\cdot\frac{1}{s(s+2)}\cdot 160\cdot\frac{s+4}{s+30}} \tag{0.66}$$

$$= \frac{(s+20)(s+30)}{s(s+2)(s+20)(s+30)+3200s+12800} \tag{0.67}$$

which has no zeros at the origin, so it is of type 0, so it cannot reject a step disturbance w with zero steady-state error. The sensitivity for feedback control is

$$\mathcal{S}_G^T = \frac{1}{1+HGD_c} = \frac{1}{1+\frac{20}{s+20}\cdot\frac{1}{s(s+2)}\cdot 160\cdot\frac{s+4}{s+30}} = \frac{s(s+2)(s+20)(s+30)}{s(s+2)(s+20)(s+30)+3200s+12800} \tag{0.68}$$

We also have

$$\left.\mathcal{S}_p^G\right|_{p=2} = -\frac{2}{s+2} \tag{0.69}$$

as before, so

$$\mathcal{S}_2^T = \mathcal{S}_G^T \mathcal{S}_2^G \tag{0.70}$$

$$= \frac{1}{1 + \frac{20}{s+20} \cdot \frac{1}{s(s+2)} \cdot 160 \cdot \frac{s+4}{s+30}} \cdot -\frac{2}{s+2} \tag{0.71}$$

$$= -\frac{2s(s+20)(s+30)}{s(s+2)(s+20)(s+30) + 3200s + 12800}, \tag{0.72}$$

which also approaches 0 as $s \to 0$.

2.8 (a) We have the following systems:

$$U = 4\left(R - Y + \frac{1}{4}x\right) = 4R - 4Y + x \tag{0.73}$$

$$x = \frac{U}{s+a} \tag{0.74}$$

$$Y = \frac{U + x}{s}. \tag{0.75}$$

Substituting the first into the second and third gives

$$x = \frac{4R - 4Y + x}{s+a} \implies x = \frac{4R - 4Y}{s+a}\left(1 - \frac{1}{s+a}\right)^{-1} = \frac{4(R-Y)}{a+s-1} \tag{0.76}$$

$$Y = \frac{4R - 4Y + 2x}{s}. \tag{0.77}$$

Plugging the second equation into the third gives

$$Y = \frac{4R - 4Y}{s} + \frac{2}{s}\left(\frac{4(R-Y)}{a+s-1}\right) = \frac{4(R-Y)(a+s+1)}{s(a+s-1)} \tag{0.78}$$

Solving for Y gives

$$Y = \frac{4R(a+s+1)}{as + 4a + s^2 + 3s + 4} \tag{0.79}$$

so the transfer function is

$$\frac{Y(s)}{R(s)} = \frac{4(a+s+1)}{as + 4a + s^2 + 3s + 4} \tag{0.80}$$

For a standard unity feedback transfer function, we have

$$T(s) = \frac{G}{1+G} \implies G = \frac{T}{1-T}, \tag{0.81}$$

so

$$G(s) = \frac{\frac{4(a+s+1)}{as+4a+s^2+3s+4}}{1 - \frac{4(a+s+1)}{as+4a+s^2+3s+4}} = \frac{4(a+s+1)}{s(a+s-1)}. \tag{0.82}$$

(b) Substituting $a = 1$, we have

$$G(s) = \frac{4(1+s+1)}{s(1+s-1)} = \frac{4(s+2)}{s^2}. \tag{0.83}$$

and $D_c(s) = 1$. Note that $GD_c(s)$ has 2 poles at the origin, so it is type 1. The error constant is

$$K_2 = \lim_{s \to 0} s^2 \frac{4(s+2)}{s^2} = 8. \tag{0.84}$$

(c) For simplicity, write $\delta \equiv \delta a$.

$$G(s) = \frac{4(1+\delta+s+1)}{s(1+\delta+s-1)} = \frac{4(\delta+s+2)}{s(\delta+s)}. \tag{0.85}$$

This has 1 pole at the origin for $\delta \neq 0$, so it is type 1. The error constant is

$$K_1 = \lim_{s \to 0} s \frac{4(\delta+s+2)}{s(\delta+s)} = 4 + \frac{8}{\delta a}. \tag{0.86}$$

2.9 (a) The $F = ma$ force law gives us

$$1000\dot{v} = 10u - 10v. \tag{0.87}$$

Taking the Laplace Transform of both sides gives

$$100sV(s) = U(s) - V(s) \implies \frac{V(s)}{U(s)} = \frac{1}{1+100s}. \tag{0.88}$$

(b) After adding the feedback loop, we have

$$V(s) = \frac{k_P}{s+0.02}\left[U(s) - V(s)\right] + \frac{0.05}{s+0.02}W(s) \tag{0.89}$$

$$\implies V(s) = \left[\frac{k_P}{s+0.02}U(s) + \frac{0.05}{s+0.02}W(s)\right]\left(1+\frac{k_P}{s+0.02}\right)^{-1} \tag{0.90}$$

$$= \frac{k_P U(s) + 0.05W(s)}{s+k_P+0.02}. \tag{0.91}$$

The error is

$$E(s) = U(s) - \frac{k_P U(s) + 0.05W}{s+k_P+0.02} = \frac{-0.05W(s) + (s+0.02)U(s)}{s+k_P+0.02}. \tag{0.92}$$

Setting $U(s) = 0$ (no input), we want to maintain an error of less than 1 m/s. If the grade is $w(t) = 2$, then $W(s) = \frac{2}{s}$, so we have

$$e_{ss} = \lim_{s\to 0}|sE(s)| = \lim_{s\to 0}\left|\frac{-0.1}{s+k_P+0.01}\right| = \frac{0.1}{k_P+0.02}. \tag{0.93}$$

If we want $e_{ss} < 1$, we want

$$k_P > \frac{2}{25} = 0.08. \tag{0.94}$$

(c) By performing an integral control, we can upgrade the order of the system, so constant grades will give 0 error. We have,

$$V(s) = \left[\frac{k_I/s}{s+0.02}U(s) + \frac{0.05}{s+0.02}W(s)\right]\left(1+\frac{k_I/s}{s+0.02}\right)^{-1} = \frac{5\cdot(20k_I U(s) + sW(s))}{2\cdot(50k_I + s(50s+1))}, \tag{0.95}$$

so

$$E(s) = U(s) - \frac{5\cdot(20k_I U(s) + sW(s))}{2\cdot(50k_I + s(50s+1))} = \frac{s(100sU(s) + 2U(s) - 5W(s))}{2\cdot(50k_I + 50s^2 + s)}. \tag{0.96}$$

Setting $U(s) = 0$ and $W(s) = \frac{2}{s}$ gives

$$E(s) = -\frac{5}{50k_I + 50s^2 + s} \tag{0.97}$$

and

$$e_{ss} = \lim_{s\to 0} sE(s) = 0, \tag{0.98}$$

as expected.

(d) Recall that

$$E(s) = \frac{s(sU(s) + 0.02U(s) - 0.05W(s))}{s^2 + 0.02s + k_I}. \tag{0.99}$$

We can compare the denominator to $s^2 + 2\zeta\omega_n s + \omega_n^2$ to get $\omega_n = \sqrt{k_I}$ and

$$\zeta = \frac{0.01}{\sqrt{k_I}}. \tag{0.100}$$

Critical damping occurs when $\zeta = 1$, so pick $k_I = 0.01^2 = 1\times 10^{-4}$.

2.10 (a) First consider

$$G(s) = \frac{0.9}{(s+0.4)(s+1.2)} = \frac{0.9}{s^2 + \frac{8s}{5} + \frac{12}{25}}. \tag{0.101}$$

Comparing the gain to $\dfrac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ We can match coefficients

$$\omega_n = \sqrt{12/25} = 0.7071 \tag{0.102}$$

$$\zeta = \frac{8/5}{2\cdot 0.7071} = 1.13138 \tag{0.103}$$

$$K = \frac{0.9}{12/25} = 1.875. \tag{0.104}$$

The rise time is given by

$$t_r = \frac{1.8}{\omega_n} < 2 \implies \omega_n > 0.9, \tag{0.105}$$

which is currently not satisfied, so we need to introduce our PI controller. To be stable, we need

$$k_I < \frac{2\zeta\omega_n(1+k_P K)}{K} = \frac{8/5\cdot(1+1.875)}{1.875} = 2.453, \tag{0.106}$$

where we let $k_P = 1$. Consider $D_c(s) = k_P + k_I/s$. The transfer function is

$$T(s) = \frac{(k_P + k_I/s)\cdot\frac{0.9}{(s+0.4)(s+1.2)}}{1+(k_P+k_I/s)\cdot\frac{0.9}{(s+0.4)(s+1.2)}} \tag{0.107}$$

$$= \frac{45(k_I+s)}{45k_I+45s+50s^3+80s^2+24s}. \tag{0.108}$$

Note that $K_I < 2.453$ such that all poles are stable. We wish to cancel out a stable pole. Choose k_I such that $s=-k_I$ is a pole, i.e.

$$45k_I - 45k_I - 50k_I^3 + 80k_I^2 - 24k_I = 0 \implies [k_I = 0,\ k_I = \frac{2}{5},\ k_I = \frac{6}{5}]. \tag{0.109}$$

We can cancel out both poles, since $k_I < 2.453$ is satisfied for both 0.4 and 1.2. Plugging in $k_I = 0.4$ gives

$$T(s) = \frac{45(0.4+s)}{45(0.4)+45s+50s^3+80s^2+24s} = \frac{0.9}{s^2+1.2s+0.9}, \tag{0.110}$$

which gives $\omega_n' = \sqrt{0.9} = 0.9487$, which satisfies $\omega_n' > 0.9$. Note that choosing $k_I = 1.2$ gives the same thing.

(b) Consider $D_c(s) = k_P + k_I/s + k_D s$. The transfer function is

$$T(s) = \frac{(k_P + k_I/s + k_D s)\cdot\frac{0.9}{(s+0.4)(s+1.2)}}{1+(k_P+k_I/s+k_D s)\cdot\frac{0.9}{(s+0.4)(s+1.2)}} \tag{0.111}$$

$$= \frac{45(k_D s^2 + k_I + k_P s)}{45k_D s^2 + 45k_I + 45k_P s + 50s^3 + 80s^2 + 24s}. \tag{0.112}$$

There is no overshoot when the system is first-order, i.e. the numerator is a factor of the denominator. We can write:

$$50s^3 + (80+45k_D)s^2 + (24+45k_P)s + 45k_I \tag{0.113}$$

$$= \frac{50s}{k_D}\left(k_D s^2 + k_I + k_P s\right) + \left(80 + 45k_D - \frac{50k_P}{k_D}\right)s^2 + \left(24 + 45k_P - \frac{50k_I}{k_D}\right)s + 45k_I \tag{0.114}$$

$$= \frac{50s}{k_D}\left(k_D s^2 + k_I + k_P s\right) + \left(\frac{80}{k_D} + 45 - \frac{50k_P}{k_D^2}\right)\left(k_D s^2 + k_I + k_P s\right) + \left(-\frac{80k_I}{k_D} + \frac{50k_P k_I}{k_D^2}\right) \tag{0.115}$$

$$+ \left(24 - \frac{50k_I}{k_D} - \frac{80k_P}{k_D} + \frac{50k_P^2}{k_D^2}\right)s. \tag{0.116}$$

The remainder of $T(s)^{-1}$ is thus

$$R(s) = \left(24 - \frac{50k_I}{k_D} - \frac{80k_P}{k_D} + \frac{50k_P^2}{k_D^2}\right)s + \left(-\frac{80k_I}{k_D} + \frac{50k_P k_I}{k_D^2}\right) \tag{0.117}$$

$$= \frac{2s(12k_D^2 - 25k_D k_I - 40k_D k_P + 25k_P^2)}{k_D^2} + \frac{10k_I(-8k_D + 5k_P)}{k_D^2}. \tag{0.118}$$

We need both terms to be zero. The constant term satisfies

$$\frac{k_P}{k_D} = \frac{8}{5} = 1.6. \tag{0.119}$$

For the linear (with respect to s) term to be zero, we require:

$$12 - 25\frac{k_I}{k_D} - 40\frac{k_P}{k_D} + 25\left(\frac{k_P}{k_D}\right)^2 = 0 \tag{0.120}$$

$$\implies 12 - 25 \cdot \frac{k_I}{k_D} - 40 \cdot \frac{8}{5} + 25 \cdot \frac{8^2}{5^2} = 0 \tag{0.121}$$

$$\implies 12 - \frac{25k_I}{k_D} = 0 \tag{0.122}$$

$$\implies \frac{k_I}{k_D} = \frac{12}{25} = 0.48. \tag{0.123}$$

We need to ensure we are cancelling out stable poles. Routh's stability criteria gives

$$k_I < \frac{(2\zeta + k_D K\omega_n)(1 + k_p K)\omega_n}{K} \tag{0.124}$$

$$= \frac{(2 \cdot 1.13138 + k_D \cdot 1.875 \cdot 0.7071)(1 + k_P \cdot 1.875) \cdot 0.7071}{1.875} \tag{0.125}$$

$$= (0.0625k_D + 0.10667)(15.0k_P + 8.0). \tag{0.126}$$

Choose $k_D = 1$ to get $k_I = 0.48$ and $k_P = 1.6$. Then we can verify that

$$(0.0625 + 0.10667)(15.0 \cdot 1.6 + 8.0) = 5.41344 > 0.48. \tag{0.127}$$