APM426: General Relativity

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Contents

1 Manifolds and Tensor Fields

1.1 Review

Note: The manifold section should serve as a review from MAT367, so we will be quickly going over it.

Definition: An n-dimensional, C^∞, real manifold M is a topological space together with a collection of subsets $\{O_\alpha\}$ satisfying the following properties:

1. $\{O_\alpha\}$ cover M.
2. For each α there is a homeomorphism $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$.
3. If any two sets O_α and O_β intersect, then $\psi_\beta \circ \psi_\alpha^{-1}$ is smooth.

Note that there are a few extra conditions (Hausdorff and paracompact), but they generally aren't important.

Let $\mathcal{F}$ denote the collection of C^∞ functions from M to $\mathbb{R}$.

Definition: Tangent vectors are maps $v : \mathcal{F} \to \mathbb{R}$ which satisfy:

1. Linearity: $v(af + g) = av(f) + g$
2. Leibniz's rule: $v(fg) = v(f)g + fv(g)$

The commutator (Lie Bracket) of two tangent vectors $[v, w] = v \circ w - w \circ v$ is also a tangent vector.

1.2 Tensors

Now, we can introduce the notion of tensors.

Definition: A (k, ℓ) tensor over a vector space V is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}$$

Some examples:

- A $(0, 1)$-tensor is a dual vector
- A $(1, 0)$ tensor is an element of V^{**}.

An interesting example is a $(1, 1)$-tensor, which is a map $V^* \times V \to \mathbb{R}$. However, we can fix $v \in V$ so $V(\cdot, v)$ is in V^{**}. But since V^{**} is canonically isomorphic to V, we have a linear map from V to V. Similarly, we can also view T as a map from $V^* \to V^*$.

Let $\mathcal{T}(k, \ell)$ be the space of all (k, ℓ)-tensors. There are two important operations on tensors:

1. **Contraction:** This is a map $C : \mathcal{T}(k, \ell) \to \mathcal{T}(k - 1, \ell - 1)$, defined by

$$CT = \sum_{\sigma=1}^{n} T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\sigma, \ldots).$$

2. **Outer product:** Given a (k, ℓ)-tensor and a (k', ℓ')-tensor, the outer product is defined by

$$(T \otimes T')(v^{1^*}, \ldots, v^{(k+k')^*}; v_1, \ldots, v_{\ell+\ell'}) = T(v^{1^*}, \ldots, v^{k^*}; v_1, \ldots, v_{\ell'})T'(v^{(k+1)^*}, \ldots, v^{(k+k')^*}; v_{\ell'+1}, \ldots, v_{\ell+\ell'}).$$

One way to construct tensors is to take the outer product of smaller tensors, i.e. vectors and dual vectors. If this is possible, then the tensor is **simple.** Let $\{v_\mu\}$ be the basis for V and $\{v^{\nu^*}\}$ its dual basis. Then,

$$\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1^*} \otimes \cdots \otimes v^{\nu_\ell^*}\}$$

forms a basis for $\mathcal{T}(k, \ell)$. Then every (k, ℓ) tensor can be written as a linear combination,

$$T = \sum_{\mu_1, \ldots, \nu_\ell = 1}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} v_{\mu_1} \otimes \cdots \otimes v^{\nu_\ell^*},$$

where the coefficients $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$ are known as the **components** of T. Note that it is often more convenient to work with just the components. For example, when applying the contraction and outer product:

- Contraction: We have
$$(CT)^{\mu_1\cdots\mu_{k-1}}{}_{\nu_1\cdots\nu_{\ell-1}} = \sum_{\sigma=1}^{n} T^{\mu_1\cdots\sigma\cdots\mu_{k-1}}{}_{\nu_1\cdots\sigma\cdots\nu_{\ell-1}}.$$
- Outer product: We have
$$(T\otimes T')^{\mu_1\cdots\mu_{k+k'}}{}_{\nu_1\cdots\nu_{\ell+\ell'}} = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} T'^{\mu_{k+1}\cdots\mu_{k+k'}}{}_{\nu_{\ell+1}\cdots\nu_{\ell+\ell'}}.$$

Theorem: The **tensor transformation law** says that
$$T'^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_\ell'} = \sum_{\mu_1,\cdots\nu_\ell=1}^{n} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x'^{\nu_\ell'}}.$$

Definition: A metric is a $(0,2)$-tensor that is also:
- Symmetric: $g_{\mu\nu} = g_{\nu\mu}$
- Nondegenerate: $g_{\mu\nu} = 0$ only if $\mu = \nu = 0$.

We can write
$$g = \sum_{\mu,\nu} g_{\nu\mu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

1.3 Abstract Index Notation

2 Curvature

How do we compare vectors in a curved space? We can't simply add or subtract since they live in different tangent spaces. To do so, we need to introduce a derivative operator.

Definition: A derivative operator ∇ on a manifold M is a map $\nabla : \mathcal{T}(k,\ell) \to \mathcal{T}(k,\ell+1)$, that satisfies the five properties:

1. Linearity: For $A, B \in \mathcal{T}(k,\ell)$ and $\alpha, \beta \in \mathbb{R}$,
$$\nabla_c(\alpha A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} + \beta B^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}) = \alpha\nabla_c(A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}) + \beta\nabla_c(B^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}).$$
2. Leibnitz Rule:
$$\nabla_e\left[A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell} B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{\ell'}}\right] = \left[\nabla_e A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}\right] B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{\ell'}} + A^{a_1\cdots a_k}{}_{b_1\cdots b_\ell}\left[\nabla_e B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{\ell'}}\right].$$
3. Commutativity with contraction:
$$\nabla_d(A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_\ell}) = (\nabla_d A)^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_\ell}$$
4. Consistent with tangent vectors. For all $f \in \mathcal{F}$ and $t^a \in V_p$, we have:
$$t(f) = t^a \nabla_a f$$
5. Torsion free:
$$\nabla_a\nabla_b f = \nabla_b\nabla_a f$$

We need to show a few important facts about this derivative operator:

- ∇ exists: To do so, pick local coordinates $\left\{\frac{\partial}{\partial x^\mu}\right\}$ and $\{\mathrm{d}x^\mu\}$. Then trhe ordinary derivative ∂_a defined by:
$$\partial_a : T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \mapsto \frac{\partial}{\partial x^\sigma} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell}$$

- The derivative operator is almost unique. Given two operators $\tilde{\nabla}_a$ and ∇_a, their difference is characterized by the tensor field C^c_{ab}, which is sometimes denoted as the **Christoffel symbol** Γ^b_{ac} when $\tilde{\nabla}_a$ is the ordinary derivative oeprator. That is,

$$\nabla_a t^b = \partial_a t^b + \Gamma^b_{ac} t^c$$

Definition: A vector v^a given at each point on the curve is said to be **parallelly transported** as one moves along the curve if the equation

$$t^a \nabla_a v^b = 0$$

is satisfied along the curve. In general, a tensor of arbitrary rank is parallelly transported if

$$t^a \nabla_a T^{b_1 \cdots b_k}{}_{c_1 \cdots c_\ell} = 0.$$

Consider a vector and choose a coordinate system. Then the above simplifies to:

$$t^a \partial_a v^b + t^a \Gamma^b_{ac} v^c = 0 \iff \frac{dv^\nu}{dt} + \sum_{\mu,\lambda} t^\mu \Gamma^\nu_{\mu\lambda} v^\lambda = 0.$$

A vector at a point p on the curve uniquely defines a parallel transported vector everywhere else on the curve. The mathematical structure arising from such a curve dependent identification of the tangent spaces of different points is called a **connection**.

Theorem: Let g_{ab} be a metric. Then there exists a unique derivative operator ∇_a satisfying $\nabla_a g_{bc} = 0$.

A direct corollary is that a metric g_{ab} naturally determines a derivative operator ∇_a. In particular, we have:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right),$$

and the coordinate basis components are

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \sum_\sigma g^{\rho\sigma} \left(\frac{\partial g_{v\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right).$$

Motivation to Curvature

Suppose we are on a riemannian manifold, i.e. the metric is positive definite. Consider a curve $\tau \in C^1([0,1] \to M^n)$, then

$$L(\tau)^2 = \int g(\tau'(t), \tau'(t))\, \mathrm{d}t\,,$$

and define the distance between x_0 and x_1 as

$$d(x_0, x_1)^2 = \inf \left\{ L(\tau)^2, \right\}$$

where $\tau(0) = x_0$ and $\tau(1) = x_1$. If τ attains this infinum, then $L(\tau + \epsilon\sigma) > L(\tau)$ for all $\epsilon > 0$ if $\sigma(0) = \sigma(1) = 0$. Then:

$$0 = \frac{d}{d\epsilon} L(\tau + \epsilon\sigma) \iff \dot{\tau} \text{ is parallel transported along } \tau,$$

where the connection is the Levi-Civita derivative,

$$\dot{\tau}^a(t) \nabla_a \dot{\tau}^b, \tag{2.1}$$

which is known as the **geodesic equation**. In coordinates, recall that $\nabla_A V^b = \partial_a V^b + \Gamma^b_{ac} V^c$, so the geodesic equation becomes

$$\frac{d\tau^\alpha}{dt} \partial_\alpha \frac{d\tau^B}{dt} + \frac{d\tau^\alpha}{dt} \Gamma^b_{\alpha\gamma} \frac{d\tau^\gamma}{dt} = 0, \tag{2.2}$$

which is sometimes written as

$$\frac{d^2\tau^B}{dt^2} + \Gamma^\beta_{\alpha\gamma}(\tau(t)) \frac{d\tau^\alpha}{dt} \frac{d\tau^\gamma}{dt} = 0. \tag{2.3}$$

If $\tau^\beta(0) = x^\beta$ and $\dfrac{d\tau^\beta}{dt}(0) = V^\beta$, then the solution (locally) is

$$\tau(t) = \exp_x(tV),$$

where the exponential map is $\exp_x : T_xM \to M, 0 \to x$, defined by:

$$(x, V) \mapsto (x, \exp_x V).$$

We can think of the exponential function as $\exp_{x_0} tv_0$ tells us to go a distance $t(v_0)_g$ in direction $\vec{v}_0$. The exponential map is smooth, and locally and smoothly defined. If we identify $(T_{x_0}M, g_{x_0}) \approx (\mathbb{R}^n, \delta_{ab})$, i.e. with the euclidean space and metric, then we can call it the **Riemannian normal coordinates at x_0,** which is a local coordinate chart at x_0. Now consider two curves $\tau(t) = \exp_x(tW)$ and $\sigma(s) = \exp_x sV$ with $\sigma(0) = \tau(0) = x$. Therefore,

$$d^2(\sigma(s), \tau(0)) = s^2, \qquad d^2(\sigma(0), \tau(t)) = t^2,$$

which is true per the geodesic equation. If we taylor expand $d^2(\tau(s), \tau(t))$ around $(s, t) = (0, 0)$, then the cross terms are zero, so

$$\begin{aligned} d^2(\sigma(s), \tau(t)) &= |sV - tW|^2 + O(|(s,t)|^3) \\ &= |sV - tW|^2 - \frac{s^2t^2}{3} R(v, w, v, w) + O(|(s,t)|^5). \end{aligned}$$

Here, R is the Riemannian curvature tensor. Therefore, curvature is just a way to describe higher order terms when computing the distance.

Formal Definition of Curvature

Now we extend to a more usual definition of curvature, which also extends to vectors that cannot be parallelly transported. Consider $f \in \mathcal{F}$ and $\omega \in \mathcal{T}(0,1)$ such that $\nabla_{[a}\nabla_{b]}f = 0$. Then,

$$\nabla_a\nabla_b(f\omega_c) = \nabla_a(\omega_c\nabla_b f + f\nabla_b\omega_c).$$

The commutator relationship is then:

$$\nabla_{[a}\nabla_{b]}(f\omega_c) = f\nabla_{[a}\nabla_{b]}\omega_c.$$

The fact that f pulls through implies that this function can only depend on the value of ω_c at p and not at any nearby points. The most general linear thing that satisfies this is some arbitrary tensor

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = R^d_{abc}\omega_d,$$

where R^d_{abc} is the Riemann curvature tensor. We can write something similar for $(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c$. Here, $V^c \in \mathcal{T}(1,0)$ and $\omega_c \in \mathcal{T}(0,1)$. Therefore, we can contract the two together to get,

$$\begin{aligned} 0 &= (\nabla_a\nabla_b - \nabla_b\nabla_a)(V^c\omega_c) \\ &= V^c R^d_{abc}\omega_d + \omega_d\nabla_{[a}\nabla_{b]}V^d. \end{aligned}$$

This gives us

$$\nabla_{[a}\nabla_{b]}V^d = -R^d_{abc}V^c.$$

By induction, we can show that if $T \in \mathcal{T}(k,\ell)$, then

$$\nabla_{[a}\nabla_{d]}T^{b_1\cdots b_k}{}_{c_1\cdots c_\ell} = -\sum_{i=1}^k R_{ade}{}^{b_i}T^{b_1\cdots e\cdots b_k}{}_{c_1\cdots c_\ell} + \sum_{j=1}^k R_{adc_j}{}^e T^{b_1\cdots b_k}{}_{c_1\cdots e\cdots c_\ell}.$$

The standard intuition behind this formulation is the failure to conserve a vector when taken under parallel transport around a closed curve.

First recall that if $U^a, B^v \in \mathcal{T}(1,0)$ and we want a vector field W such that $[U,V]^c = W^c$, then

$$\begin{aligned} W(f) &= U(V(f)) - V(U(f)) \\ &= U^b\nabla_b(V^a\nabla_a f) - V^a\nabla_a(U^b\nabla_b f) \\ &= (U^a\nabla_a V^c - V^a\nabla_a U^c)\nabla_c f \end{aligned}$$

Consider a surface S. Let's attempt to parallel transport V^a around a parallelogram $(0,0) \to (\Delta s, 0) \to (\Delta s, \Delta t) \to (0, \delta t)$. Let us fix $\omega_b \in \mathcal{T}(0,1)$ and let us see how $V^a\omega_a$ changes.

Define $S^a = \left(\frac{\partial}{\partial s}\right)^a$ and $T^b = \left(\frac{\partial}{\partial t}\right)^b$ be coordinate tangent vectors to S. Note that $[S,T]=0$. Then,

$$\begin{aligned}\delta_1 &= (\Delta s)\frac{d}{ds}(v^a\omega_a)\Big|_{(\Delta s/2,0)} + O((\Delta s)^3)\\ &= (\Delta s)S^b\nabla_b(v^a\omega_a)\Big|_{(\Delta s/2,0)}\\ &= (\Delta s)S^b v^a \Delta b\omega_a\Big|_{(\Delta s/2,0)}\end{aligned}$$

Similarly,

$$\delta_3 = -(\Delta s)S^b v^a\nabla_b\omega_a\Big|_{(\Delta s/2,\Delta t)}.$$

However, the v^a in this last expression is the vector we get after being transported halfway across the parallelogram. But first, note that

$$\delta_1+\delta_2+\delta_3+\delta_4 = O((\Delta s)^2+(\Delta t)^2),$$

so since the difference between the vectors at different points vary by second order, we can effectively ignore them. Therefore,

$$\delta_1+\delta_2+\delta_3+\delta_4 = v_{\text{new}} - v_0 = -S^bT^cR^d_{bca}v_0$$

The Riemann tensor has some symmetries,

1. ${R_{abc}}^d = -{R_{bac}}^d$
2. ${R_{[abc]}}^d = 0$.
3. If $\nabla_a g_{bc} = 0$, then $R_{abcd} = -R_{abdc}$.
4. $\nabla_{[e}{R_{ab]c}}^d = 0$ is the **Bianchi** identity.

We can prove these,

1. ${R_{abc}}^d\omega_d = \nabla_{[a}\nabla_{b]}\omega_c$
2. Consider some form $T_{[abc]}$. Then
$$\nabla_{[a}T_{bcd]}$$
is a form. But taking the derivative again
$$\nabla_{[a}\nabla_bT_{cde]} = 0.$$
To see this, we can rewrite
$$\nabla_{[a}T_{bcd]} = \partial_aT_{bcd} + \Gamma^e_{[ab]}T_{ecd}+\cdots.$$
Since the Γ are symmetric, then the Γ terms disappear, and we get ∂_aT_{bcd}. Therefore,
$$\nabla_{[a}\nabla_bT_{cde]} = \partial_{[a}\partial_bT_{cde]} = 0.$$
3.

2.1 Geodesics

Let $\Sigma^{n+1}\subseteq M^n$ be an $(n-1)$-dimensional submanifold of M, i.e. a hypersurface. Then,

$$\mathbb{R}^{n-1}\approx T_p\Sigma\subseteq T_pM\approx\mathbb{R}^n.$$

There are three cases for g

$$g\Big|_{T_p\Sigma\times T_p\Sigma} > 0 \iff \Sigma \text{ is spacelike at } p,$$

which is alwys true for Riemannian metrics. For Lorentizn metrics,

$$\det g|_{(T_p\Sigma)^2} = 0 \iff \Sigma \text{ is null at } p$$

$$g|_{(T_p\Sigma)^2} < 0 \iff \Sigma \text{ is timelike at } p.$$

We claim that there exists a nonzero normal vector $N \in T_pM$ such that $g(N, X) = 0$ for all $X \in T_p\Sigma$. Note that N won't be null, i.e. $N \notin T_p\Sigma$ (except possibly in case 2). Therefore WLOG,

$$g(N, N) = \pm 1$$

except of course in case 2.

Gaussian Normal Coordinates near Σ Let $\overline{X} = (x^1, \dots, x^{n-1})$ be any coordinates on a neighbourhood of p in Σ. Then,

$$(x^1, \dots, x^{n-1}, t) \in \mathbb{R} \mapsto \exp_{\overline{X}=(x^1,\dots,x^{n-1})} tN \in M$$

gives coordinates in a neighborhood of Σ. Notice that

$$N^a N_a = \pm 1$$

and for all $X^b \in T_q\Sigma_0$, we have

$$g(N, X) = N_a X^a = 0.$$

Claim: The geodesic $t \in \mathbb{R} \to \exp_{\underline{X}} tN$ remains orthogonal to Σ_t for all small t.

Proof. The tangent vectors

$$X_i^a = \left(\frac{\partial}{\partial x^i}\right)^a$$

for $i \in \{1, \dots, n-1\}$ form a coordinate basis for $T_p\Sigma_t$, and $N^b = \left(\frac{\partial}{\partial t}\right)^b$ denotes the tangent to the geodesic. Recall,

$$\left[N, X_{(i)}\right]^a = 0.$$

This is equivalent to

$$T^b \nabla_b X^a = X^b \nabla_b T^a,$$

for $X \in \{X_1, \dots, x_{n-1}\}$. We then claim that

$$N^a X_a = 0,$$

for all $|t| \ll 1$. Its derivative along the geodesic is

$$\begin{aligned} N^b \nabla_b (N^a X_a) &= \cancel{(N^b \nabla_b N^a)} X_a + N^a N^b \nabla_b X_a \\ &= N_a X^b \nabla_b N^a \\ &= \frac{1}{2} X^b \nabla_B (N_a N^a) \\ &= \frac{1}{2} X^b \nabla_B (\pm 1) \\ &= 0. \end{aligned}$$

□

2.2 Geodesic Deviation Equation and Jacobi Fields

Let $t \in I \subseteq \mathbb{R} \to \sigma(t) \in M$ be a geodesic. Now consider a surface

$$(s, t) \in B_1(0) \subseteq \mathbb{R}^2 \to \sigma_s(t) = \sigma(t; s) \in M$$

formed by geodesics $t \in I_s \subseteq \mathbb{R} \to \sigma_s(t)$, i.e.

$$\dot{\sigma}_s^a \nabla_a \dot{\sigma}_s = 0.$$

If we have a geodesic, we can re-parametrize it. That is, $\sigma(bt + c)$ is also a geodesic. If σ depends on s, then $\sigma_s(b(s)t + c(s))$ is also a geodesic. We have some freedom to choose $b(s)$ and $c(s)$. Let us choose $b(s)$ such that

$$\dot{\sigma}^a \dot{\sigma}_a = \pm 1.$$

We can also choose $c'(0)$ so that

$$\dot{\sigma}^a X_a \Big|_{\sigma_0(0)} = 0.$$

Finally, we can choose $X^a = \left(\frac{\partial}{\partial s}\right)^a$ and $c(s)$ such that

$$\dot{\sigma}^a X_a \bigg|_{\sigma_s(0)} = 0.$$

As before, $\dot{\sigma}^b \nabla_b(\dot{\sigma}^a X_a) = 0$. Therefore, X^a remains orthogonal to $\dot{\sigma}$ along $\sigma_0(t)$ for all t, which is a result we've seen before.

We can linearize the geodesic equation around $\sigma_0(t)$ to get a linear 2nd order equation for X^a. That is, we can compute the velocity

$$\begin{aligned} v^a &= \dot{\sigma}^b \nabla_b X^a \\ a^a &= \dot{\sigma}^c \nabla_c v^a \end{aligned}$$

of a nearby geodesic relative to $\sigma_0(t)$. We can rewrite,

$$a^a = \dot{\sigma}^c \nabla_c \left(\dot{\sigma}^b \nabla_b X^a\right),$$

which includes two derivatives. Recall that the order of derivatives matter and if we want to switch the order, we need to include curvature. Note that since $[\dot{\sigma}, X] = 0$, we can rewrite,

$$\begin{aligned} a^a &= \dot{\sigma}^c \nabla_c \left(X^b \nabla_b \dot{\sigma}^a\right) \\ &= \dot{\sigma}^c (\nabla_c X^b) \nabla_b \dot{\sigma}^b + X^b \dot{\sigma}^c \nabla_c \nabla_b \dot{\sigma}^a \\ &= (X^c \nabla_c \dot{\sigma}^b) \nabla_b \dot{\sigma}^a + X^b \dot{\sigma}^c \nabla_b \nabla_c \dot{\sigma}^a - R^a_{cbd} \dot{\sigma}^d X^b \dot{\sigma}^c \\ &= -R^a_{cbd} \dot{\sigma}^d X^b \dot{\sigma}^c. \end{aligned}$$

Note that a is linear in X, so this is the linear second order equation we wanted. In coordinates, this becomes,

$$\frac{d^2}{dt^2} X^\alpha + R^\alpha_{\gamma\beta\delta} \dot{\sigma}^\gamma \dot{\sigma}^\delta X^\beta = 0.$$

The initial conditions are $X^\alpha(0) = X_0^\alpha$ and $\dot{X}^\alpha(0) = V_0^\alpha$. There are n choices for both, so $2n$ degrees of freedom. Note that 2 of them correspond to the affine reparametrization of $\sigma_0(t)$.

Solutions $X^\alpha(t)$ along $\sigma_c(t)$ are called **Jacobi Fields.**

2.3 Computing the Riemann Tensor

How do we compute R^d_{abc}? There are different methods to do so, but we begin with the coordinate method. Given a tensor field $\omega_d \in \mathcal{T}(0,1)$, then

$$\begin{aligned} \frac{1}{2} R^d_{abc} \omega_d &= \frac{1}{2} (\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c \\ &= \nabla_{[a} \nabla_{b]} \omega_c. \end{aligned}$$

In local coordinates,

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c{}_{ab} \omega_c.,$$

and we can write it in a more useful form,

$$D_b \omega_c = \partial_b \omega_c - \Gamma^d{}_{bc} \omega_d.$$

We have,

$$\begin{aligned} \nabla_a \nabla_b \omega_c &= \partial_a (\partial_b \omega_c - \Gamma^d{}_{bc} \omega_d) - \Gamma^e{}_{ab} \left(\partial_e \omega_c - \Gamma^d{}_{ec} \omega_d\right) - \Gamma^e{}_{ac} \left(\partial_b \omega_\ell - \Gamma^d_{be} \omega_d\right) \\ \implies \nabla_{[a} \nabla_{b]} \omega_c &= -\partial_{[a} \Gamma^d{}_{b]c} \omega_d + \Gamma^e{}_{c[a} \Gamma^d{}_{b]e} \omega_d. \end{aligned}$$

Therefore,

$$R_{\alpha\beta\gamma}{}^\delta = -\frac{\partial}{\partial x^\alpha} \Gamma^\delta{}_{\beta\gamma} + \frac{\partial}{\partial x^\beta} \Gamma^\delta{}_{\alpha\gamma} + \Gamma^\epsilon{}_{\gamma\alpha} \Gamma^\delta{}_\epsilon - \Gamma^\epsilon{}_{\gamma\beta} \Gamma^\delta{}_{\alpha\epsilon}.$$

There are different types of curvature tensors. We have the **Ricci curvature,**

$$R_{ac} = R^b_{abc}$$

and the scalar curvature $R = R^a_a = R_{ac} g^{ca}$.

2.4 Twice Contracted Identity

Starting from the Bianchi identity, we can expand it:

$$0 = 2\nabla_a R_{bcd}{}^e + 2\nabla_b R^e_{cad} + 2\nabla_c R^e_{abd}.$$

Contracting on c, e gives

$$\nabla_a R_{bd} - \nabla_b R_{ad} + \nabla_e R^e_{abd},$$

and contracting it by ad (i.e. multiplying by g^{ad}) gives

$$\begin{aligned} 0 &= \nabla^a R_{ba} - \nabla_b R_a{}^a + \nabla^e R_{be} \\ &= 2\left(\nabla^a R_{ba} - \frac{1}{2}\nabla_b R\right) \\ &= 2\nabla^a \left(R_{ab} - \frac{1}{2} R g_{ab}\right), \end{aligned}$$

where $G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}$ is often known as the **Einstein tensor**, and the twice contracted identity tells us that it is divergence free, which gives us a conservation law.

Recall that we can write,

$$\begin{aligned} 0 &= \nabla_a v^a \\ &= \frac{\partial v^a}{\partial x^a} + \Gamma^a{}_{ab} v^b. \end{aligned}$$

We can write out an explicit formula for the contracted Christoffel symbols:

$$\begin{aligned} \Gamma^a{}_{ab} &= \frac{1}{2} g^{ad}\left(\frac{\partial g_{db}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{bd}}{\partial x^d} + g^{ad}\frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^a}\right) \\ &= \frac{1}{2} g^{ac}\frac{\partial}{\partial x^b} g_{ac}. \end{aligned}$$

Alternatively,

$$\begin{aligned} \Gamma^a{}_{ab} &= \frac{1}{2}\frac{\partial}{\partial x^b}\log|g| \\ &= \frac{1}{2}\frac{\partial}{\partial x^b}\sum_{i=1}^{n}\log|\lambda_i| \\ &= \frac{1}{2}\frac{1}{\lambda_i}\frac{\partial \lambda^i}{\partial x^b} \\ &= \frac{1}{2} g^{ac}\frac{\partial}{\partial x^b} g_{ac} \end{aligned}$$

where $|g| = |\det g_{ij}|$.

2.5 Differential Forms

Let ∇_a be a connection on M. We can define the derivative operator

$$d : \Lambda^p(M) \to \Lambda^{p+1}(M)$$

by

$$\omega_{a_1 \dots a_p} \mapsto \nabla_{[b}\omega_{a_1 \dots a_p]}$$

which we can expand to

$$\nabla_{[b}\omega_{a_1 \dots a_p]} = \frac{\partial}{\partial x^i}\omega_{a_1 \dots a_p} + \sum_{i=1}^{p} C^c_{[ba_i}\omega_{a_1 \dots |c| \dots a_p}.$$

But this last term is zero, so d only depends on the differential topology of the manifold, and not its differential geometry. It is independent of the connection choice! Therefore, we often denote the derivative of ω as $d\omega$.

Note that $\omega \in \Lambda^p(M)$ is **closed** when $\mathrm{d}\omega = 0$ and ω is **exact** when it can be written as $\omega = \mathrm{d}\eta$.

The manifold M is **orientable** if and only if there exists an $\epsilon \in \Lambda^n(M)$ that is continuous non-vanishing. If $\alpha \in \Lambda^n(M)$ and M is oriented (by ϵ) then

$$\int_M \alpha$$

is defined locally in charts and globally by partitions of unity, and is independent of coordinates on the chart.

However, things become spicy once we bring a metric into play. A pseudo-Riemannian metric g on M^n selects a preferred volume form ϵ (up to a sign (orientable)). This is to ensure

$$\epsilon^{a_1 \dots a_n}\epsilon_{a_1 \dots a_n} = (-1)^s n!$$

Many nice things follow from this. In right-handed coordinates on $U \subseteq M$, we have

$$\epsilon \to \sqrt{|g|}\,\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n \,,$$

where $|g| = |\det g_{\alpha\beta}|$ in some coordinates. Let:

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right),$$

and consider the change of basis

$$\frac{\partial}{\partial x^{\bar\mu}} = \frac{\partial x^\mu}{\partial x^{\bar\mu}}\frac{\partial}{\partial x^\mu} = \Lambda^\mu_{\bar\mu}\frac{\partial}{\partial x^\mu}.$$

Then,

$$\begin{aligned} g_{\bar\mu\bar\nu} &= \frac{\partial x^\mu}{\partial x^{\bar\mu}}\frac{\partial x^\nu}{\partial x^{\bar\nu}} g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \\ &= \Lambda^\mu_{\bar\mu}\Lambda^\nu_{\bar\nu} g_{\mu\nu}, \end{aligned}$$

where the Λ are Jacobians. Then,

$$|\bar g|^{1/2} = |\det \Lambda||g|^{1/2},$$

is how the determinant of the metric changes wrt coordinates.

allows for the reduction to the Standard Stoke's Theorem via integration by parts.

Lemma 1: If the derivative ∇_a and volume form ϵ are compatible with g_{ba} then U is a subdomain $U \subset\subset M^n$ implies

$$\int_U f(\nabla_a V^a)\epsilon = \int_{\partial U}(f V^a n_a)\,\mathrm{d}A - \int_U (\nabla_a f)V^a_a \epsilon,$$

where n_a is an outer normal form[a]. If $f \in C^1(U)$ and $V \in \mathcal{T}(1,0)$ is c^1.

[a] For example if $U = \{x \in M, \phi(x) < 0\}$ with $|d\phi| = 1$, then we can identify $n = \mathrm{d}\phi$

Proof. Assume a coordinate chart ψ covers U. Then, we can integrate,

$$\begin{aligned} -\int_U f(\nabla_a V^a)\epsilon &= -\int_{\psi(U)} f\left(\frac{\partial V^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\beta}V^\beta\right)\sqrt{g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= -\int_{\psi(U)} f\left(\frac{\partial V^\alpha}{\partial x^\alpha}\sqrt{g} + \frac{V^\beta}{|g|^{1/2}}\frac{\partial |g|^{1/2}}{\partial x^\beta}\right)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= -\int_{\psi(U)} f\frac{\partial V^\alpha}{\partial x^\alpha}|g|^{1/2} + fV^\beta\frac{\partial |g|^{1/2}}{\partial x^\beta}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \,, \\ &= +\int_{\psi(U)} V^\alpha\frac{\partial}{\partial x^\alpha}(f\sqrt{g}) + \frac{\partial(fV^\alpha)}{\partial x^\alpha}\sqrt{g}\mathrm{d}^n x - 2\int_{\partial\psi(U)} fV^\alpha\sqrt{g}n_\alpha \mathrm{d}^{n-1}x \\ &= \int_{\psi(U)}\frac{\partial}{\partial x^\alpha}(V^\alpha f\sqrt{g}) + (\partial_\alpha f)V^\alpha\sqrt{g}\mathrm{d}^n x - 2\int_{\partial\psi(U)} fV^\alpha\sqrt{g}n_\alpha \mathrm{d}^{n-1}x. \end{aligned}$$

Which gives

$$-\int_{\partial\psi(U)}(f\sqrt{g}+V^{\alpha}n_{\alpha})\mathrm{d}^{n-1}x+\int_{\psi(U)}(\partial_{\alpha}f)V^{\alpha}\sqrt{g}\,\mathrm{d}x^{n}\,,$$

which is the same thing as what we want in our lemma.

Note that the reason there is an extra $\sqrt{g}$ term is because of the **minkowski capacity**,

$$\text{Area}(\Sigma)=\lim_{\epsilon\to 0}\frac{vol(U_{\epsilon}-U_{0})}{\epsilon},$$

and the volume will have a $\sqrt{g}$ factor, so we need to add this to the area when we work with coordinates. □

3 Stress Energy Tensor

In Euclidean geometry $\mathbb{R}^3$, we have $h_{\mu\nu}=\text{diag}(1,1,1)$, which implies that $\Gamma^{\mu}_{\alpha\beta}=0$, so the covariant derivative agrees $\nabla_{\mu}=\partial_{\mu}$, geodesics are straight lines, and parallel transport is curve independent. In $\mathbb{R}^4$ in special relativity, we have

$$\eta_{\mu\nu}=\text{diag}(-1,1,1,1),$$

and the Christoffel symbols vanish, etc. Although there is no inertial frame, there is still a future direction and orientation. Consider the $(M^n,g^{\mu\nu})$ Lorentzian manifold:

- At each point, we have a lightcone, and we can decide the future direction in a consistent way (orientable).
- Massive particles, absent other forces, follow timelike geodesics, parametrized such that

$$g(\sigma'(s),\sigma'(s))=-1$$

- The 4-velocity is

$$U^a=\frac{d\sigma^a}{ds}$$

  and the 4-momentum is

$$P^a=mU^a.$$

- The energy of our particle, measured by a non-comoving observer at the same point in spacetime M, is

$$\tilde{U}^b p_b=g_{ab}(\tilde{U}^b,p^a),$$

  where $\tilde{U}^b$ is the 4-velocity of observer.
- $\tilde{U}$ and U makes an angle of θ with each other. Note that

$$\tanh\psi=v=\tan\theta,$$

  so we can write the energy as

$$\tilde{U}^b p_b=m\cosh\psi=\frac{m}{\sqrt{1-|v|^2}},$$

  where $|v|$ is the relative 3-velocity.
- If $g_{ab}\tilde{X}^a\tilde{U}^b=0$, then the momentum in the $\tilde{X}^a$ direction is $g_{ab}\tilde{X}^a p^b=\dfrac{mv}{\sqrt{1-|v|^2}}$.

Something else that is affected by boosts is number density. Consider N particles at rest and draw some box with volume r^3 in the x,y,z direction. The density is

$$n=\frac{N}{r^3}.$$

In a moving reference frame, the Lorentz boost, the particles are now moving at some relative velocity $(v,0,0)$ in the x-direction. The density now becomes

$$\bar{n}=\frac{n}{\gamma r^3},$$

due to length contraction, so density transforms like the first component of a 4-vector. We can treat this as a 4-vector, where the other 3 directions are the fluxes. The **number flux vector** is

$$\begin{aligned} N^a &\xrightarrow{\text{comoving}} (n,0,0,0) \\ &\xrightarrow{\text{x-boosted}} (n/\sqrt{1-|v|^2}, \frac{nv}{\sqrt{1-|v|^2}}, 0, 0) \\ &\xrightarrow{\text{boosted}} \left(n/\sqrt{1-|v|^2}, \frac{nv^1}{\sqrt{1-|v|^2}}, \frac{nv^2}{\sqrt{!-|v|^2}}, \frac{nv^3}{\sqrt{1-|v|^2}}\right). \end{aligned}$$

Note that the number of particles is observer independent, i.e.

$$n^2 = g_{ab}(N^a, N^b),$$

where n is the scalar rest density and is the first component of N^a in comoving coordinates.

This motivates the question of what the energy density of these particles might look like. The energy density is the energy per particle times the density of the particles, i.e.

$$\begin{aligned} &\xrightarrow{\text{comoving}} mn \\ &\xrightarrow{\text{x-boosted}} \frac{mn}{1-|v|^2}. \end{aligned}$$

This transforms like a $(2,0)$ tensor $T^{ab} = P^a \otimes N^b$. This motivates us to write,

$$\begin{aligned} T^{ab} &\xrightarrow{\text{comoving}} \begin{pmatrix} mn & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{\text{x-boosted}} \begin{pmatrix} \frac{mn}{1-v^2} & \frac{mvn}{1-v^2} & 0 & 0 \\ \frac{mvn}{1-v^2} & \frac{mnv^2}{1-v^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

which is the energy momentum tensor for **dust.** More generally, T^{ab} is the **stress-energy tensor** of matter, with the following properties

- The energy-density observed by observer having velocity v is

$$T_{ab}v^a v^b$$

- If $g_{ab}(X^a, V^b) = 0$, Then the energy flux of matter in X^b direction is

$$T_{ab}v^a X^b$$

  and

$$T_{ab}X^a v^b$$

  is the X^a momentum density, and

$$T_{ab}X^a Y^b$$

  is the X^a momentum flux in the Y^b direction, where $g_{ab}(X^a, Y^b) = g_{ab}(V^a, Y^b) = 0$.

Lemma 2: The stress-energy tensor is symmetric, i.e. $T^{ab} = T^{ba}$.

Proof. Note that T^{a0} is the X^a momentum density and T^{0a} is the energy flux in the x^a direction. These are the same quantity, so

$$T^{a0} = T^{0a}.$$

To show that $T^{ab} = T^{ba}$ for $a, b \neq 0$, we can use a rotation argument. Consider small volume cuboid elements. The torque in one direction is created by contributions from four sides. For example, the z torque is given by

$$\left(T^{xy}\Big|_{y=-r} r^2 + T^{xy}\Big|_{y=r} r^2 \right) - \left(T^{yx}\Big|_{x=-r} r^2 + T^{yx}\Big|_{x=r} r^2 \right).$$

The zz momentum of inertia scales like r^5. The angular acceleration is a ratio, and to prevent diverging for small r, we evaluate at $y = 0 + r^3 \approx 0$ instead, and we get

$$T^{xy} = T^{yx}$$

in order to prevent infinite acceleration. □

3.1 Perfect Fluids

Perfect fluids have no viscosity (no heat conduction), so there is no shear force. Therefore, $T^{\alpha\beta} = 0$ for all $\alpha \neq \beta \in \{1, 2, 3\}$ spacelike indices. Since there is no heat conduction, heat (energy density) can only be transported in the direction of the fluid, i.e.

$$T^{\alpha 0} = 0 \forall \alpha \in \{1, 2, 3\}.$$

In a special relativity frame, we have

$$T^{ab} \xrightarrow{SR} \begin{pmatrix} T^{00} & 0 & 0 & 0 \\ 0 & T^{11} & 0 & 0 \\ 0 & 0 & T^{22} & 0 \\ 0 & 0 & 0 & T^{33} \end{pmatrix}$$
$$\to \begin{pmatrix} \rho(x,y,z,t) & 0 & 0 & 0 \\ 0 & P(x,y,z,t) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

The stress energy tensor is also divergence free, which is the statement that energy and momentum is conserved locally. Note that in special relativity,

- T_{00} is the energy density
- T_{0i} is the energy flux in the i direction
- T_{i0} is the i-momentum density
- T_{ij} is the flux of momentum in the j direction.

Consider an imaginary cube. The flux through the 6 faces of the cube is given by

$$\frac{\partial}{\partial t}\Big|_{(0,0,0)} (T^{00}L^3) = -\left(T^{01}\Big|_{(L,0,0)} - T^{01}\Big|_{(0,0,0)} \right) \frac{L^2}{L} - \left(T^{02}\Big|_{(0,L,0)} - T^{02}\Big|_{(0,0,0)} \right) \frac{L^2}{L} - \left(T^{03}\Big|_{(0,0,L)} - T^{03}\Big|_{(0,0,0)} \right) \frac{L^2}{L},$$

which gives, after taking the limit $L \to 0$,

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x} + \frac{\partial T^{02}}{\partial y} + \frac{\partial T^{03}}{\partial z} = 0, \tag{3.1}$$

which is the equation for conservation of energy. Similarly, conservation of momentum can be written as

$$\frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{j1}}{\partial x} + \frac{\partial T^{j2}}{\partial y} + \frac{\partial T^{j3}}{\partial z} = 0. \tag{3.2}$$

More generally, we can write that

$$\nabla_a T^{ab} = 0.$$

Note that we can write

$$T^{ab} = (\rho + P)U^a U^b + P g^{ab}$$
$$T_{ab} = (\rho + P)U_a U_b + P g_{ab}.$$

Taking the covariant derivative, we have

$$\begin{aligned} 0 = \nabla_a T^{ab} &= U^b \nabla_a(\rho U^a) + \rho U^a \nabla_a U^b + P\left(U^b \nabla_a U^a + U^a \nabla_a U^b\right) + (g^{ab} + U^a U^b)\nabla_a P \\ &= U^b\left(\nabla_a(\rho U^a) + P\nabla_a U^a\right) + (\rho + P)U^a \nabla_a U^b + (g^{ab} + U^a U^b)\nabla_a P. \end{aligned}$$

Contracting with U^b (note: $U^b U_b = -1$) in order to get the motion in a particular direction,

$$0 = -1\left[\nabla_a(\rho U^a) + P\nabla_a U^a\right] + \cancel{\frac{1}{2}(\rho + P)U^a \nabla_a(U^b U_b)} + \cancel{(U^a - U^a)}\nabla_a P,$$

which gives us

$$\begin{aligned} 0 &= \nabla_a(\rho U^a) + P\nabla_a U^a = U^a \nabla_a \rho + (\rho + P)\nabla_a U^a \\ 0 &= (\rho + P)U^a \nabla_a U^b + (g^{ab} + U^a U^b)\nabla_a P. \end{aligned}$$

How do we interpret this? In SR, $g_{ab} = \eta_{ab}$ and $\nabla_a = \partial_a$, and $\rho \gg P$ for $|v| \ll 1$. Then,

$$0 = \frac{\partial \rho}{\partial t} + (\vec{v} \cdot \vec{\nabla})\rho + \rho\left(\vec{\nabla} \cdot \vec{v}\right) = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}),$$

which gives us the **continuity equation.** We can also recover Netwon's second law, for $b = 1, 2, 3$.

$$\rho\left(\frac{\partial U^b}{\partial t} + (\vec{v} \cdot \vec{\nabla})U^b\right) = -\partial_b P.$$

Together, they form the 3D compressible Euler equations for fluids. We have five unknowns here, ρ, U^1, U^2, U^3, P, and four equations. We can get a fifth equation by getting an equation of state (i.e. ideal gas law).

Idea: For dust, we have $P = 0$, which gives

$$\begin{aligned} 0 &= \nabla_a(\rho U^a) \\ 0 &= \rho U^a \nabla_a U^a. \end{aligned}$$

The first condition is the continuity condition (density is transported by velocity), and the second condition tells us that the dust particles follow geodesics.

3.2 Klein-Gordon Wave Equation

For $\phi : M \to \mathbb{R}$, the Klein-Gordon wave equation is

$$\nabla^a \nabla_a \phi - m^2 \phi = 0,$$

which can be written as

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\phi - m^2\phi = 0.$$

Sometimes, we write

$$-\Box = -\frac{\partial^2}{\partial t^2} + \Delta.$$

This comes from conservation of energy/momentum frmo the following stress energy tensor.

$$T^{ab} = \nabla^a \phi \nabla^\phi - \frac{1}{2} g^{ab}\left(\nabla^c \phi \nabla_c \phi + m^2 \phi^2\right)$$

Contracting it with ∇_a gives

$$\begin{aligned} 0 = \nabla_a T^{ab} &= (\nabla_a \nabla^a \phi)\nabla^b \phi + \nabla_a \phi(\nabla_a \nabla^a \phi) - \frac{1}{2} g^{ab}\left(\nabla_a \nabla^c \phi \nabla_c \phi + \nabla^c \phi \nabla_a \nabla_c \phi + 2m^2 \phi \nabla_a \phi\right) \\ 0 &= (\nabla_a \nabla^a \phi - m^2 \phi)\nabla^b \phi + \nabla^a \phi(\nabla_a \nabla^b \phi) - \frac{1}{2}\left(\nabla^b \nabla_a \phi \nabla^a \phi + \nabla^c \phi \nabla^b \nabla_c \phi\right). \end{aligned}$$

By the torsion free condition, we get the desired wave equation.

3.3 Maxwell's Equations

We can write the **Faraday tensor** as

$$F_{ab} = F_{[ab]} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Maxwell's equations are very simple in this notation. Namely,

$$\begin{aligned} \nabla^a F_{ab} &= -4\pi J_b \\ \nabla_{[a} F_{bc]} &= 0, \end{aligned}$$

where J^b is the charge/current density four-vector. First, we can check that J^b is divergence free. Note that

$$\begin{aligned} \nabla^b J_b &= -\frac{1}{4\pi}\nabla^b\nabla^a F_{ab} \\ &- \frac{1}{4\pi}\partial_a\partial_b F^{ab} \\ &= 0, \end{aligned}$$

which tells us that current is the flux of the charge density. The second relationship tells us that locally, we can write

$$F_{ab} = \nabla_a A_b - \nabla_b A_a,$$

where A is a four-vector, known as the **vector potential.** Plugging this into the first relationship gives

$$\nabla^a\nabla_a A_b - \nabla_b \nabla_a A^a - R^a_{abc} A^c = -4\pi J_b.$$

Notice that we can write $A_a = A'_a + \nabla_a \chi$ (choosing the Lorentz gauge). Then we can show, with some work, that

$$\nabla_a A^a = \nabla_a A'^a + \nabla_a \nabla^a \chi.$$

We can choose $\nabla_a\nabla^a\chi = \nabla_a A^a$ to make $\nabla_a A'^a = 0$. Therefore, our conservation law gives

$$\nabla^a\nabla_a A_b - R_{bc}A^c = -4\pi J_b.$$

How will charges move in these fields? The answer is that the acceleration is given by

$$u^a\nabla_a u^b = \frac{q}{m} F^b_c u^c,$$

and corresponds to Newton's second law.

3.4 Lorentz Gauge

In special relativity, the Lorentz gauge seeks solutions

$$A_b = C_b e^{iS(t,x,y,z)},$$

where C^b is a constant, i.e. parallel transport. Plugging this into the wave equation in the absence of charges, i.e. $\nabla^a\nabla_a A_b = 0$. Then,

$$\begin{aligned} \partial_a A_b &= A_b i \partial_a S \\ \partial^a\partial_a A_b &= A_b \left(-\partial^a S \partial_a S + i\partial^a\partial_a S\right). \end{aligned}$$

For this to vanish, both the real and imaginary components need to vanish, i.e.

$$\partial^a S \partial_a S = 0 = \partial^a \partial_a S.$$

Similarly, $\nabla_b A^b = 0$ gives

$$C^b \nabla_b S = 0.$$

Let $k_a = \nabla_a S$. Then $k_a k^a = 0$ is a null vector orthogonal to C^b. Then the surfaces

$$\Sigma_\lambda := \{x : S(x) = \lambda\}$$

are null surfaces. In fact k^a is both normal and tangent to Σ and its integral curves. In particular, $\nabla_a S$ is a vector field and its integral curves $\phi(t,x)$ solve

$$\begin{cases} \dfrac{\partial \phi^\alpha}{\partial s} = g^{\alpha\beta}\nabla_\beta S\left(\phi(s,x)\right) \\ \phi^\alpha(s,x) = x^\alpha. \end{cases}$$

To show that they stay on the same level set, we can plug ϕ into S and take the derivative. We can compute,

$$\begin{aligned} \frac{d}{ds}S(\phi(s,x)) &= \nabla_a S \frac{\partial \phi^\alpha}{\partial s} \\ &= \nabla_\alpha S g^{\alpha\beta}\nabla_\beta S \\ &= 0. \end{aligned}$$

Moreover, these integral curves are null geodesics, because

$$\begin{aligned} 0 &= \nabla_b(\nabla_a S \nabla^a S) \\ &= 2\nabla_b \nabla_a S \nabla^a S \\ &= 2\nabla^a S \nabla_a \nabla_b S \\ &= 2\nabla^a S \nabla_a \nabla^c S \\ &= 2k^a \nabla_a k^c \end{aligned}$$

where the third line follows from torsion free-ness, and

$$k^c = g^{ca}\nabla_a S$$

is tangent to the integral curve

$$ad - k_c k^c = \nabla_a S \nabla^a S = 0.$$

3.5 Relating Geometry to Physics

For a manifold (M^n, g_{ij}), there is a tensor (Einstein tensor)

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R,$$

that satisfies $\nabla^a G_{ab} = 0$. But from physics, we know there is another tensor that satisfies

$$\nabla^a T_{ab} = 0.$$

For Maxwell, this tensor is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_a^c - g_{ab}F_{cd}F^{cd}\right),$$

where $\nabla^a T_{ab} = 0$ if $J_b = 0$. Note that more generally, we need to take the stress energy of charges + currents to get the more general statement.

The brilliant idea of Einstein was that maybe these are the same things. The Einstein Field equations says geometry = physics, and says

$$G_{ab} = \alpha T_{ab},$$

for some constant α. We want to show that $\alpha = 8\pi$. To show this, we can compare the geodesic variation equation (which gives us the acceleration of a nearby geodesic)

$$U^a \nabla_a (U^b v_b X^d) = -R_{abc}{}^d U^a U^c X^b,$$

to Newton's second law, which gives

$$F = ma = -(\vec{X} \cdot \vec{\nabla})\vec{\nabla}\phi,$$

where the potential ϕ is $\Delta\phi = 4\pi\rho$, which can be rewritten as

$$F = -X^b \partial_b \partial^d \phi,$$

where $\partial_b \partial^d$ is the **Hessian**, and also has trace $4\pi\rho$. The Hessian is a part of the Riemann tensor $R_{abc}{}^d$. Contracting by $R^b{}_d$ gives

$$4\pi\rho = \Delta\phi = R_{ac} U^a U^c,$$

so we can *guess* that

$$R_{ac} = 4\pi T_{ac},$$

but the right side is divergence free. We can rearrange

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$
$$R_{ac} = G_{ac} - \frac{!}{2} G g_{ac},$$

Noting that $R_{ab} U^a U^b \sim 4\pi\rho$, we get

$$4\pi\rho = (G_{ab} - \frac{G}{2} g_{ab}) U^a U^b$$
$$\alpha \left(T_{ab} - \frac{T}{2} g_{ab} \right) U^a U^b$$
$$= \alpha \left(\rho + T/2\right)$$
$$= \frac{\alpha}{\rho} 2,$$

which eventually gives us $\alpha = 8\pi$. Therefore, Einstein's equation is

$$G_{ab} = 8\pi T_{ab}.$$

4 Physics

4.1 Linearized Gravity

Suppose we are given T_{ab}. Then we can find g_{ab}. Recall that,

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\nabla_b g_{cd} + \nabla_c g_{bd} - \nabla_d g_{bc} \right),$$

where the derivatives are linear in g, but multiplying it by g^{ad} makes it nonlinear. Furthermore,

$$R_{abc}{}^d = -\partial_{[a} \Gamma_{b]c}{}^d + \Gamma^e{}_{c[a} \Gamma_{b]e}{}^d.$$

These are very hard equations to solve as they are nonlinear. One of the first things we'll do is **linearize gravity** (i.e. weak gravity). We take,

$$g_{ab} = \eta_{ab} + \gamma_{ab},$$

where η_{ab} is Minkowski metric and γ_{ab} is some small variation. Its solutions correspond to **gravitational waves.** We can write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \lambda \gamma_{\alpha\beta} + \mathcal{O}(\lambda^2).$$

More generally, given $(M, g_{ab}(s))$ and a 1-parameter group

$$\phi_s : M \to M$$

of diffeomorphisms, then $(M, g_{ab}(s))$ and $(M, \phi_s^*(g_a b(s)))$ will present equivalent theories. Therefore, we can (without loss of generality), set

$$\gamma_{ab} := \left. \frac{\partial g_{ab}}{\partial s} \right|_{s=0}.$$

and

$$\tilde{\gamma}_{ab} = \left. \frac{\partial}{\partial s} \right|_{s=0} \left(\phi_s^*(g_a b(s)) \right)$$
$$= \left. \frac{\partial g_{ab}}{\partial s} \right|_{s=0} + \mathcal{L}_{-v} g_{ab}$$
$$= \gamma_{ab} - \mathcal{L}_{+v} g_{ab}.$$

Now, we can compute,

$$\mathcal{L}_v g_{ab} = v^c \nabla_c g_{ab} + g_{ad} \nabla_b v^d + g_{db} \nabla_a v^d.$$

Using the Levi-Civita connection, we have $v^c \nabla_c g_{ab} = 0$, which gives

$$\tilde{\gamma}_{ab} = \gamma_{ab} - \nabla_b v_a - \nabla_a v_b.$$

This says that in linearized gravity, if we take any γ_{ab} and disturb it to get $\tilde{\gamma}_{ab}$, then we get an equivalent theory, as expected. Therefore, we pick

$$g_{ab} = \eta_{ab} + \gamma_{ab},$$

where to first-order in γ_{ab}, we have

$$g^{ab} = \eta^{ab} - \gamma^{ab},$$

and

$$\begin{aligned} \Gamma^c{}_{ab} &= \frac{!}{2} \eta^{cd} (\partial_a \gamma_{db} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) \\ \Gamma^c{}_{cb} &= \frac{1}{2} (\partial^c \gamma_{cb} + \partial_b \gamma - \partial^c \gamma_{cb}) = \frac{1}{2} \partial_b \gamma, \end{aligned}$$

where $\gamma = \gamma^a_a$. Then, we have

$$R_{acb}{}^d = -2\partial_{[a} \Gamma^d_{c]b}$$

and

$$\begin{aligned} R_{ab} = R^c_{acb} &= \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb} \\ &= \frac{1}{2} (\partial^c \partial_a \gamma_{cb} + \partial^a \partial_b \gamma_{ac} - \partial^c \partial_c \gamma_{ab} - \partial_a \partial_b \gamma), \end{aligned}$$

so

$$\partial^c \partial_c \gamma = \Box \gamma = \left(-\frac{\partial^2}{\partial t^2} + \Delta \right) \gamma.$$

4.2 Homogeneous Isotropic Universe Models

For a homogeneous, isotropic universe, we have the **FLRW tensor**,

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \times \begin{cases} \mathrm{d}\Psi^2 + \sin^2 \Psi (\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2) & k = 1 \\ \mathrm{d}\Psi^2 + \Psi^2 (\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2) & k = 0 \\ \mathrm{d}\Psi^2 + \sinh^2 \Psi (\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2) & k = -1 \end{cases}, \tag{4.1}$$

where k is the curvature. For the $k = 0$ case, we can compute,

$$\begin{aligned} R_{\tau\tau} &= -3 \frac{\ddot{a}}{a} \\ R_{xx} &= a\ddot{a} + 2\dot{a}^2 \\ R_{**} &= \frac{a\ddot{a} + 2\dot{a}^2}{a^2}, \end{aligned}$$

We can compute the Ricci tensor to be,

$$R = -R_{\tau\tau} + 3R_{**} = 6 \left(\frac{\ddot{a} + \dot{a}^2}{a^2} \right). \tag{4.2}$$

The einstein tensor is given by,

$$\begin{aligned} G_{ab} &= R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} = 8\pi \left((\rho + P) \, \mathrm{d}\tau_a \, \mathrm{d}\tau_b + P g_{ab} \right) \\ G_{\tau\tau} &= -\frac{3\ddot{a}a}{a^2} + 3 \left(\frac{\ddot{a}a + \dot{a}^2}{a^2} \right) = \frac{3\dot{a}^2}{a^2} = 8\pi\rho \\ G_{**} &= \frac{a\ddot{a} + 2\dot{a}^2}{a^2} - 3 \left(\frac{\ddot{a}a + \dot{a}^2}{a^2} \right) = 8\pi P. \end{aligned}$$

This gives two equations,

$$\begin{aligned}\dot{a}^2 &= \frac{8\pi}{3}\rho a^2 - k \\ \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}\left(\rho + 3P\right),\end{aligned}$$

where the k is the curvature. From the second equation, we immediately know that a is concave. This is important as it means in future time, the universe will either continue expanding, or expand and then contract. We can also go back in time, and see that there will be some time such that $a = 0$. Hubble's Law tells us that

$$\frac{dR}{d\tau} = \frac{R}{a}\frac{da}{d\tau} = RH(\tau). \tag{4.3}$$

Recall that we only care about dust, in which $P = 0$ and for radiation, where $P = \rho/3$, and density is a function of time. Let's integrate! The derivative of the first equation is

$$\begin{aligned}2\ddot{a}\dot{a} &= \frac{8\pi}{3}\left(\dot{\rho}a^2 + 2\rho a\dot{a}\right) \\ \implies -\frac{8\pi}{3}a\dot{a}\left(\rho + 3P\right) &= \frac{8\pi}{3}\left(\dot{\rho}a^2 + 2\rho a\dot{a}\right) \\ \implies 0 &= \dot{\rho}a^2 + 3a\dot{a}(\rho + P).\end{aligned}$$

For dust, this simplifies to

$$0 = \frac{1}{a}\frac{d}{d\tau}(\rho a^3) \tag{4.4}$$

and for radiation, we have,

$$0 = \frac{1}{a}\frac{d}{d\tau}(\rho a^4). \tag{4.5}$$

This tells us that,

- Dust: $\rho a^3 = C_1$
- Radiation $\rho a^4 = C_2$

So ρ and a are inversely proportional (to some exponent). Therefore, radiation energy dominates in the early universe and as $a(\tau) \to \infty$, dust energy will dominate. Therefore, we have,

$$\dot{a}^2 = \frac{C_q}{a^q} - k, \tag{4.6}$$

where $q \in \{1, 2\}$ depending on whether we have dust or radiation. Note that if $k \leq 0$, we have a continuous expansion, $\dot{a} \neq 0$ and after a long time, the rate of expansion will approach $-k$. If $k = 1$, then the universe expands to a maximum a_c, then reverses the process ending in a big crunch.

If $k = 0$ we can solve this to get

$$a(\tau) = \left(\frac{q+2}{2}\sqrt{C_q}\tau\right)^{2/(q+2)},$$

which scales like $\tau^{2/3}$ for dust and $\tau^{1/2}$ for radiation. For $k = 1$, we have

- Dust: $a(\tau) = \frac{c_1}{2}(1 - \cos\eta)$
- Radiation: $a(\tau) = \sqrt{c_2}\left[1 - \left(1 - \frac{\tau}{\sqrt{c_2}}\right)^2\right]^{1/2}$

4.3 Hubble's Law

Photons follow null geodesics with tangent vector n^b. Suppose that $k = 0$. Then, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are killing vector fields. Without loss of generality, assume

$$n_b\left(\frac{\partial}{\partial y}\right)^b\bigg|_{\Sigma_d} = 0 = n_b\left(\frac{\partial}{\partial z}\right)^b\bigg|_{\Sigma_1}. \tag{4.7}$$

Since the dot product of n_b with a tangent vector of a Killing field is preserved, we have

$$n_b \left(\frac{\partial}{\partial y}\right)^b \bigg|_{\Sigma_2} = 0 = n_b \left(\frac{\partial}{\partial z}\right)^b \bigg|_{\Sigma_2},$$

and $n_b \left(\frac{\partial}{\partial x_b}\right)^b$ is proportional to the energy. Since we have photons, the dot product using the time-like vector and the space-like vector givs the same thing. Including the proportionality factor, we have

$$E = \frac{1}{a(\tau)} \cdot n_b \left(\frac{\partial}{\partial x_b}\right)^b = \frac{1}{\sqrt{g}} \cdot n_b \left(\frac{\partial}{\partial x_b}\right)^b.$$

For our photon,

$$\omega(\tau)a(\tau) = n_b \left(\frac{\partial}{\partial x}\right)^b = \text{const},$$

or

$$\frac{a(\tau_2)}{a(\tau_1)} = \frac{\omega(\tau_1)}{\omega(\tau_2)} = \frac{\lambda_2}{\lambda_1}.$$

So if a photon has a small wavelength (and high energy) when the universe was small, then it will have a long wavelength (and a low energy) when the universe is big. This describes redshift, and is quantified by

$$\begin{aligned} z &= \frac{\lambda_2}{\lambda_1} - 1 \\ &= \frac{a(\tau_2)}{a(\tau_1)} - 1 \\ &\approx \frac{a(\tau_1) + a'(\tau_1)(\tau_2 - \tau_1)}{a(\tau_1)} - 1 \\ &= \frac{a'(\tau_1)}{a(\tau_1)}(\tau_2 - \tau_1) \\ &= H(\tau)R, \end{aligned}$$

where R is the distance between nearby galaxies, since we used a Taylor approximation.

4.4 Particle Horizons

This prompts the question: Looking back in time, can we see the whole big bang, or just part of it?

Lightcones are invariant under conformal factors. Conformal factors are functions of the metric that preserve the dot product. So for $k = 0$, the FLRW metric has some the same lightlike surfaces as Minkowski space. Then the question becomes whether the integral

$$t = \int_0^\tau \frac{\mathrm{d}s}{a(s)}$$

diverges or converges. If it diverges, then going to a finite τ (i.e. in Minkowski space) gives us to an infinite t (i.e. allowing us to see the whole big bang). If it converges, then we can only see a finite part of the universe's past. This converges for both dust and radiation. We can only see stuff inside the **particle horizon**, and this grows as time progresses. Therefore, as time progresses, we can see more and more of the universe's past.

It turns out that if $k = 1$ and we are in a universe dominated by dust, then we can see the big bang at the maximum size of the universe. If we are in a universe dominated by radiation, we can see the big bang at the big crunch.

4.5 Schwarzschild Solution

The Newtonian potential for a point source is

$$u(x) = -\frac{m}{|x|^{n-2}}$$

in n dimensions, where $\Delta u = 0$. We seek a vaccuum solution to Esintein's equation

$$R_{ab} = 8\pi T_{ab} = 0,$$

(except maybe at a point). We are also seeking a solution with a lot of symmetries.

- We call (M^a, g_{ab}) **stationary** if g_{ab} admits a Killing vector field ξ^c.

  This implies that we have time translation symmetry, so $t \to t + \delta t$ should give the same metric.

- It is **static** if in addition, ξ^c is hypersurface orthogonal ($\xi_{[a}\nabla_b\xi_{c]}$).

  This implies time reflection $t \to -t$ symmetry.

- It is **spherically symmetric** if $SO(3)$ generates a group of isometries on it.

  The spherically symmetric condition tells us that there are three other spatial killing vector fields, so the orbit of x traces out a two-dimensional sphere S^2 on the hyperplane at $t = 0$ with a radius

  $$r \propto \sqrt{A}.$$

  They also rule out cross terms like $\mathrm{d}r\,\mathrm{d}\theta$ because if their components were nonzero, then the projection onto the 2-sphere would have a nonzero direction, but the action of $SO(3)$ rotation on any point gives a 2-sphere with no preferred direction.

The metric is therefore

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta a\,\mathrm{d}\phi^2\right).$$

Tetrad methods: Choose an orthonormal basis for each T_pM for each $p \in M$. Consider an orthonormal basis $\tilde{e}_\mu = (e_\mu)^a$ for $\mu \in \{0, 1, 2, 3\}$. Consider,

$$\tilde{\omega}_{\mu\nu} \equiv \omega_{a\mu\nu} \equiv (e_\mu)^b \nabla_a (e_\nu)_b,$$

which are known as the **Connection 1-forms.** By orthonormal, we mean that

$$(e_\mu)^a (e_\nu)_a = \eta_{\mu\nu}.$$

and we can write,

$$\delta_b^a = \eta^{\mu\nu}(e_\mu)^a)e_\nu)_b.$$

Proof. Contract with $(e_\sigma)^b$ to get

$$\begin{aligned}(e_\sigma)^a &= \eta^{\mu\nu}(e_\mu)^a \eta_{\nu\sigma} \\ &= (e_\sigma)^a.\end{aligned}$$

□

The connection 1-forms are a lot like the Christoffel symbols but are antisymmetric in terms of its indices, i.e.

$$\tilde{\omega}_{\mu\nu} = -\tilde{\omega}_{\nu\mu}.$$

The Riemann tensor can be written as,

$$\tilde{R}_\mu^\nu = R_{ab\mu}^\nu = \mathrm{d}\tilde{\omega}_\mu^\nu + \mathrm{d}\tilde{\omega}_\mu^\sigma \wedge \tilde{\omega_\sigma}^\nu,$$

where $\eta_{\mu\nu}$ is used to raise and lower greek indices. The derivative of the basis 1-forms is given by

$$\mathrm{d}\tilde{e}_\mu = \tilde{e}_\sigma \wedge \tilde{\omega_\mu}^\sigma.$$

Idea: We have not justified why these are true, but if we assume they are, we can use the above two formulas to find what the connection 1-form is, and what the Riemann curvature tensor is. We will try to justify the formula

$$\tilde{R}_\mu^\nu = \mathrm{d}\tilde{\omega}_\mu^\nu + \tilde{\omega}_\mu^\sigma \wedge \tilde{\omega}_\sigma^\nu.$$

Recall that

$$\begin{aligned}R_{ab\mu\nu} &= R_{abc}{}^d (e_\mu)^c (e_\nu)_d \\ &= (e_\mu)^c (\nabla_a\nabla_b - \nabla_b\nabla_a)(e_\nu)_c.\end{aligned}$$

The first term can be written as,

$$\begin{aligned}(e_\mu)^c\nabla_a\nabla_b(e_\nu)_c &= \nabla_a\left((e_\mu)^c\nabla_b(e_\nu)_c\right) - \nabla_a(e_\mu)^c + \nabla_b(e_\nu)_c \\ &= \nabla_a(\omega_{b\mu\nu}) - \eta^{\alpha\beta}[\nabla_a(e_\mu)_f](e_\alpha)^c(e_\beta)^f v_b(e_\nu)_c \\ &= \nabla_a(\omega_{b\mu\nu}) - \eta^{\alpha\beta}\omega_{a\beta\mu}\omega_{b\alpha\nu}.\end{aligned}$$

where we used $\delta^c_f = \eta^{\alpha\beta}(e_\alpha)^c(e_\beta)_f$ to get the second equality. Combining the first and the second term (which is taking the anti-symmetry), we can identify the first term as $\mathrm{d}tilde\omega^\nu_\mu$ and the second term as the wedge product.

Let us choose,

$$(\tilde{e}_0)_a = f^{1/2}_{(r)}(\tilde{\mathrm{dt}})_a$$

such that

$$\begin{aligned}\tilde{e}_0 &= \sqrt{f(r)}\tilde{\mathrm{dt}} \\ \tilde{e}_1 &= \sqrt{h(r)}\tilde{\mathrm{dr}} \\ \tilde{e}_2 &= r\tilde{\mathrm{d}\theta} \\ \tilde{e}_3 &= r\sin\theta\tilde{\mathrm{d}\phi},\end{aligned}$$

and their derivatives are

$$\begin{aligned}\mathrm{d}\tilde{e}_0 &= \frac{1}{2}f^{-1/2}f'\tilde{\mathrm{dt}}\wedge\tilde{\mathrm{dt}} \\ \mathrm{d}\tilde{e}_1 &= 0 \\ \mathrm{d}\tilde{e}_2 &= \mathrm{d}r\wedge\mathrm{d}\theta \\ \mathrm{d}\tilde{e}_3 &= \sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi + r\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\,.\end{aligned}$$

Using our formula for the derivative of basis 1-forms gives

$$\frac{1}{2}f^{-1/2}f'\tilde{\mathrm{dr}}\wedge\tilde{\mathrm{dt}} = h^{1/2}\,\mathrm{d}r\wedge\tilde{\omega}^1_0 + r\,\mathrm{d}\theta\wedge\tilde{\omega}^2_0 + r\sin\theta\,\mathrm{d}\phi\wedge\tilde{\omega}^3_0.$$

One natural guess could be that $\tilde{\omega}^2_0 = \tilde{\omega}^3_0 = 0$ and

$$\tilde{\omega}^1_0 = \alpha_1\,\mathrm{d}r + \frac{1}{2}\frac{f'}{(fh)^{1/2}}\,\mathrm{d}t\,,$$

which would cause the first term to match, and the last two terms to be zero. Let's hope this is in agreement when we apply the formula on the other terms. We get,

$$\begin{aligned}0 &= -e_0\wedge\omega^1_0 + e_2\wedge\omega^2_1 + e_3\wedge\omega^3_1 \\ \implies 0 &= -f^{1/2}\,\mathrm{d}t\wedge\omega^1_0 + r\,\mathrm{d}\theta\wedge\omega^2_1 + r\sin\theta\,\mathrm{d}\phi\wedge\omega^3_1.\end{aligned}$$

Letting $\alpha_1 = 0$ and setting

$$\begin{aligned}\omega^2_1 &= \alpha_2\,\mathrm{d}\theta + \alpha_3\,\mathrm{d}\phi \\ \omega^3_1 &= \frac{\alpha_3}{\sin\theta}\,\mathrm{d}\theta + \alpha_4\,\mathrm{d}\phi\end{aligned}$$

makes the second equation true. For $\mathrm{d}\tilde{e}_2$, we have

$$\mathrm{d}r\wedge\mathrm{d}\theta = -f^{1/2}\,\mathrm{d}t\wedge\omega^2_0 - h^{1/2}\,\mathrm{d}r\wedge\omega^2_1 + r\sin\theta\,\mathrm{d}\phi\wedge\omega^3_2.$$

Substituting in the guesses we already made, we arrive at $\alpha_2 = \dfrac{1}{h^{1/2}}$. To make this equation true, we can guess

$$\omega^3_2 = -\frac{\alpha_3 h^{1/2}}{r\sin\theta}\,\mathrm{d}r + \alpha_5\,\mathrm{d}\phi\,.$$

Going to the last equation, we have,

$$\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi + r\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi = f^{1/2}\,\mathrm{d}t\wedge\omega^0_3 - h^{1/2}\,\mathrm{d}r\wedge\omega^3_1 - r\,\mathrm{d}\theta\wedge\omega^3_2.$$

Plugging everything in, we get $\alpha_3 = 0$ and $\alpha_4 = -\frac{\sin\theta}{h^{1/2}}, \alpha_5 = -\cos\theta$. In conclusion, we have our six connection 1-forms,

$$\begin{aligned}
\omega_0^2 &= 0\\
\omega_0^3 &= 0\\
\omega_0^1 &= \frac{1}{2}\frac{f'}{(fh)^{1/2}}\,\mathrm{d}t\\
\omega_1^2 &= -\frac{1}{h^{1/2}}\,\mathrm{d}\theta\\
\omega_1^3 &= -\frac{\sin\theta}{h^{1/2}}\,\mathrm{d}\phi\\
\omega_2^3 &= -\cos\theta\,\mathrm{d}\phi
\end{aligned}$$

We can now compute the Riemann tensors. Since $R_{ab\mu\nu} = -R_{ab\nu\mu}$, we only need to compute the ones where $\mu < \nu$. Therefore,

$$\begin{aligned}
\tilde{R}_0^1 &= \mathrm{d}\omega_0^1 + \omega_0^\sigma \wedge \omega_\sigma^0 = \frac{1}{2}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right)\mathrm{d}r \wedge \mathrm{d}t\\
\tilde{R}_0^2 &= \mathrm{d}\omega_0^2 + \omega_0^\sigma \wedge \omega_\sigma^2 = \omega_0^1 \wedge \omega_1^2 = -\frac{1}{2}\frac{f'}{f^{1/2}h}\,\mathrm{d}t \wedge \mathrm{d}\theta\\
\tilde{R}_0^3 &= \mathrm{d}\omega_0^3 + \omega_0^\sigma \wedge \omega_\sigma^3 = \omega_0^1 \wedge \omega_1^3 = -\frac{1}{2}\frac{f'\sin\theta}{f^{1/2}h}\,\mathrm{d}t \wedge \mathrm{d}\phi\\
\tilde{R}_1^2 &= \mathrm{d}\omega_1^2 - \omega_1^3 \wedge \omega_2^3 = \frac{1}{2}h^{-3/2}h'\,\mathrm{d}r \wedge \mathrm{d}\theta\\
\tilde{R}_1^3 &= \mathrm{d}{\omega_1}^3 + \omega_1^2 \wedge \omega_2^3 = -cos\theta h^{-1/2}\,\mathrm{d}\theta \wedge \mathrm{d}\phi + \frac{1}{2}\sin\theta h^{-3/2}h'\,\mathrm{d}r \wedge \mathrm{d}\phi + \cos\theta h^{-1/2}\,\mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{1}{2}\sin\theta h^{-3/2}h'\,\mathrm{d}r \wedge \mathrm{d}\phi\\
\tilde{R}_2^3 &= \mathrm{d}\omega_2^3 - \omega_1^2 \wedge \omega_1^3 = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi - \frac{\sin\theta}{h}\,\mathrm{d}\theta \wedge \mathrm{d}\phi = \left(1 - \frac{1}{h}\right)\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi\,.
\end{aligned}$$

We are looking for vacuum solutions, so the Ricci tensor needs to vanish. We have,

$$\begin{aligned}
R_{00} &= {R_{010}}^1 + {R_{020}}^2 + {R_{030}}^3 = \frac{1}{2}\frac{1}{(fh)^{1/2}}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right) + \frac{1}{2}\frac{f'}{f^{1/2}h} - \frac{f'\sin\theta}{rfh}\\
R_{11} &= {R_{101}}^0 + {R_{121}}^2 + {R_{121}}^3 = -\frac{1}{2}\frac{1}{(fh)^{1/2}}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right) + \frac{1}{2}\frac{h'}{h^2 r}\\
R_{22} &= R_{33} = {R_{202}}^0 + {R_{212}}^1 + {R_{232}}^3 = -\frac{1}{2}\frac{f'}{rfh} + \frac{1}{2}\frac{h'}{rh^2} + \frac{1}{r^2}\left(1 - \frac{1}{h}\right),.
\end{aligned}$$

Note that we have to evaluate $\tilde{R}_0^1 = {R_{ab0}}^1$ through its orthonormal basis. That is,

$$\begin{aligned}
\mathrm{d}t \wedge \mathrm{d}r &= \frac{1}{(fh)^{1/2}}e_0 \wedge e_1\\
\mathrm{d}t \wedge \mathrm{d}\theta &= \frac{1}{rf^{1/2}}e_0 \wedge e_2\\
\mathrm{d}r \wedge \mathrm{d}\theta &= \frac{1}{h^{1/2}r}e_1 \wedge e_2\\
\mathrm{d}r \wedge \mathrm{d}\phi &= \frac{1}{h^{1/2}r\sin\theta}e_1 \wedge e_3\\
\mathrm{d}\theta \wedge \mathrm{d}\phi &= \frac{1}{r^2\sin\theta}e_2 \wedge e_3
\end{aligned}$$

We want the trace to be zero, so each of the $R_{ii} = 0$. We have three equations and two unknowns, so we hope that they are equivalent with each other! Adding the first two equations, we get

$$\frac{f'}{f} + \frac{h'}{h} = 0 \implies \frac{d}{dr}\ln(fh) = 0 \implies f(r)h(r) = K,$$

for some constant K. By rescaling time we can make $K = 1$. Therefore, $h = 1/f$ and $h' = -\frac{1}{f^2}f'$. The third equation gives,

$$0 = -f'r + (1 - f) \implies (rf)' = 1.$$

Therefore,

$$f = 1 + \frac{C}{r},$$

and we have derived the **Schwarzschild metric,**

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{4.8}$$

where we identified $C = -2M$, derived such that when M is small, we get Newtonian gravity. Note that at $r = 2M$, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial r}$ are not linearly independent. At first, Schwarzschild thought that this was no good since it is singular at this point, but it turns out that we just need different coordinates to describe $r = 2M$.

A true physical singularity occurs at $r = 0$ (i.e. physical, curvature tensor blows up).

Outside the spherical mass distribution, such as a star, we have a vaccuum solution, so $G_{ab} = 0 = R_{ab}$. Inside the star, the Ricci tensors are

$$\begin{aligned}
R_{00} &= \frac{1}{2}(fh)^{-1/2}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right) + \frac{f'}{rfh} \\
R_{11} &= \frac{1}{2}(fh)^{-1/2}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right) + \frac{h'}{rh^2} \\
R_{22} = R_{33} &= -\frac{1}{2}\frac{f'}{rfh} + \frac{1}{2}\frac{h'}{rh^2} + \frac{1-h^{-1}}{r^2},
\end{aligned}$$

and the scalar curvature is

$$\begin{aligned}
R &= -R_{00} + R_{11} + R_{22} + R_{33} \\
&= -(fh)^{-1/2}\frac{d}{dr}\left(\frac{f'}{(fh)^{1/2}}\right) - 2\frac{f'}{rfh} + 2\frac{h'}{rh^2} + 2\frac{(1-h^{-1})}{r^2}.
\end{aligned}$$

Assume the star is a perfect fluid, so the Einstein equation gives

$$\begin{aligned}
8\pi\rho &= G_{00} = \frac{h'}{rh^2} + \frac{1-h^{-1}}{r^2} = \frac{1}{r^2}\left(r\left(1 - \frac{1}{h}\right)\right)' \\
8\pi P &= G_{11} \\
8\pi P &= G_{22} = G_{33}.
\end{aligned}$$

The solution to the first equation gives

$$\begin{aligned}
r\left(1 - \frac{1}{h(r)}\right) &= 8\pi\int_0^r \rho(r)r^2\,\mathrm{d}r \\
\implies r\left(1 - \frac{1}{h(r)}\right) &= 2m(r) \\
\implies h(r)^{-1} &= 1 - \frac{2m(r)}{r}.
\end{aligned}$$

Note that outside the star for $\rho = 0$, this agrees with the Schwarzschild solution. For $r \le R$, the proper mass is

$$M_p = \int_0^R \rho(r)r^2\left(1 - \frac{2m(r)}{r}\right)^{-1/2}\mathrm{d}r > m(R) = M.$$

Letting $f = e^{2\Phi}$, we get

$$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3}{r(r - 2m(r))} \sim \frac{m(r)}{r^2},$$

so we get the Newtonian potential in the limit.

4.6 Geodesics of Schwarzschild

Determining the geodesics may seem to be extremely complex, but because of the high degree of symmetries. Let $\sigma(\tau)$ be a proper time (or affinely parametrerized) geodesic. Then

$$-K = \langle \dot{\sigma}, \dot{\sigma} \rangle_g$$

where $K \in \{0, 1\}$. We have the symmetry $\theta \leftrightarrow \pi - \theta$, allowing us to write

$$\theta(0) = \pi/2, \qquad \dot{\theta}(0) = 0, \qquad \theta(\tau) = \frac{\pi}{2}.$$

We also have our Killing fields. The conserved quantities are

$$\begin{aligned} E &= -\langle \dot{\sigma}, \frac{\partial}{\partial t} \rangle_g = \dot{t}\left(1 - \frac{2M}{r}\right) \\ L &= \langle \dot{\sigma}, \frac{\partial}{\partial \phi} \rangle = \dot{\phi} r^2, \end{aligned}$$

and using these conservations, we can write

$$\begin{aligned} -K = \langle \dot{\sigma}, \dot{\sigma} \rangle_g &= -\left(1 - \frac{2M}{r}\right) + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + \cancel{r^2\dot{\theta}^2} + r^2 \dot{\phi}^2 \\ &= -\left(1 - \frac{2M}{r}\right) \frac{E^2}{(1 - \frac{2M}{r})^2} + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + \frac{L^2}{r^2}. \end{aligned}$$

We can write,

$$\frac{E^2}{2} = \frac{1}{2}\left(K + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) + \frac{\dot{r}^2}{2},$$

where the first term is the potential energy, i.e.

$$\begin{aligned} V(r) &= \left(\frac{1}{2} - \frac{M}{r}\right)\left(K + \frac{L^2}{r^2}\right) \\ &= \frac{K}{2} - \frac{MK}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}, \end{aligned}$$

where the second term is the Newtonian potential, second term corrects for angular momentum, and the last term is the Einstein correction. We have two cases,

- Case 1: $K = 1$. We get

  $$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}.$$

  We can find its critical points by setting $V'(r) = 0$, which gives

  $$0 = \frac{1}{r^4}\left(Mr^2 - L^2 r + 3ML^2\right),$$

  which gives

  $$\begin{aligned} R_\pm &= \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M} \\ &= \frac{M}{2}\left(\frac{L^2}{M}\right)\left(1 \pm \sqrt{1 - 12\left(\frac{M}{L}\right)^2}\right), \end{aligned}$$

  so there can be two circular orbits for a given angular momentum L, given that $\frac{L^2}{M^2} > 12$, where one of them is the standard Newtonian stable one, and the other one is closer to the central body and unstable.

  We have,

  $$\begin{aligned} \frac{E^2(R_\pm)}{2} &= V(R_\pm) \\ &= \frac{1}{2} - \frac{M}{R_\pm} \end{aligned}$$

5 Appendix C: Maps of Manifolds

5.1 Pushforwards and Pushbacks

In this appendix, we will talk about maps of manifolds. If we have a map $\phi \in C^\infty(M, N)$, i.e. $\phi : M^m \to N^n$, then tangent vectors to M can be pushed forward through ϕ to N, and cotangent vectors to N can be pulled back through ϕ to M (unless ϕ is a diffeomorphism).

A tangent vector $v \in T_x M$ pushes forward to

$$\phi^* v \in T_{\phi(X)} N.$$

Note that Wald uses ϕ^* for pushforward and ϕ_* for pullback, which is the opposite of the standard notation. If $f \in C^\infty(N)$ then $f \circ \phi \in C^\infty(M)$. Then we also have

$$(\phi^* v)(f) = v(f \circ \phi).$$

Similarly, if $\mu \in T_{\phi(x)} N$, then its pullback

$$\phi_* \mu \in T_x M$$

is

$$(\phi_* \mu)_a (v^a) := \mu_a (\phi^* v)^a,$$

for all $v \in T_\lambda M$. More generally for arbitrary tensors we can push forward any tensor with only up indices

$$(\phi^* T)^{a_1 \cdots a_k} (\mu_1)_{a_1} \cdots (\mu_k)_{a_k} = T^{a_1 \cdots a_k} (\phi_* \mu_1)_{a_1} \cdots (\phi_* \mu_k)_{a_k},$$

or pull-back any tensor with only down indices Mixed index tensors T^a_b present a problem unless $\phi : M \to N$ is a diffeomorphism, in which case

$$(\phi^* T)^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} v^{b_1} \cdots v^{b_l} \mu^1_{a_1} \cdots \mu^k_{a_k} := T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} (\phi_x \mu^1)_{a_1} \cdots \left(\left(\phi^{-1}\right)^k v_\ell \right)^{b_\ell},$$

which can also be interpreted as the Jacobian map from the coordinates on M to the coordinates on N.

Note that if $\phi : M \to M$ and $(\phi^* g)_{ab} = g_{ab}$, then ϕ is called an **isometry.** More generally if $\phi^* T = T$ for some tensor field T, then ϕ is called a **symmetry** of T. That is, if we have a physical theory on $(M, T(1), \ldots, T_{(k)})$ and another physical theory on $(M', T'_{(1)}, T'_{(k)})$, then they are equivalent (or same) if there is a diffeomorphism $\phi : M \to M'$, which pushforwards each T_{ij} to T'_{ij}.

5.2 Lie Derivatives

Recall (at least on a compact manifold) a vector field v^a on a compact manifold M induces a 1-parameter group of diffeomorphisms

$$\phi_s : M \to M$$

given by

$$\begin{cases} \dfrac{\partial \phi_s(x)}{\partial s} = v(\phi_s(x)) \\ \phi_0(x) = x, \end{cases}$$

locally in s. We can define the Lie derivative of a tensor field

$$T \in \mathcal{T}(k, \ell)$$

on M (in direction v) by

$$(\mathcal{L}_v T)^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} = \lim_{s \to 0} \frac{(\phi^*_{-s} T)^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} - T^{a_1 \cdots a_k}{}_{b_1 \cdots b_e ll}}{s},$$

and it also satisfies the Leibnitz rule.

Lemma 3: Some properties of the Lie derivative:

(a) If $T = f \in C^\infty(M)$, then

$$\mathcal{L}_v f = v(f)$$

(b) If $T = w \in \mathcal{T}(1, 0)$ is a vector field, then

$$\mathcal{L}_v w = [v, w]$$

Proof. 1. Build a coordinate system on M where s is the first coordinate. Then,

$$\phi_{-s}(x_1, \cdots, x_n) = (x_1 - t, x_2, \cdots, x_n).$$

The Jacobian of this change of variables is

$$(\phi_{-t}^*)^\mu_\nu = \delta^\mu_\nu,$$

so

$$(\phi_{-t}^* T)^{\mu_1 \cdots \mu_k}{}_{v_1 \cdots v_\ell} = T^{\mu_1 \cdots \mu_k}{}_{v_1 \cdots v_k}(x_1 + t, x_2, \cdots, x_n).$$

Plugging this back, we have

$$\mathcal{L}_v(f) = \lim_{s \to 0} \frac{f(x_1 + s, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{s} = v(f),$$

is the standard directional derivative.

2. In these coordinates,

$$(\mathcal{L}_{\sqsubseteq} \mathcal{T})^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = \frac{\partial T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}{\partial x^1,}$$

so

$$\begin{aligned}(\mathcal{L}_\nu w)^\mu &= \frac{\partial w^\mu}{\partial x^1} \\ &= v^1 \frac{\partial w^\mu}{\partial x^1} \\ &= \sum_\alpha v^\alpha \frac{\partial w^\mu}{\partial x^\alpha} - w^\alpha \frac{\partial v^\mu}{\partial x^\alpha} \\ &= [v, w]^\mu,\end{aligned}$$

where we recall that $v^\nu = \left(\frac{\partial}{\partial x}\right)^\mu$.

□

Once we can define the Lie derivative of functions and vector fields, we can define the Lie derivative of any tensor field through duality.

Corollary 1:
(a) If $\mu \in \mathcal{T}(0, 1)$, then

$$(\mathcal{L}_\nu \mu)_a = v^c \nabla_c \mu_a + \nu_c \nabla_a \nu^c$$

(b) If $T \in \mathcal{T}(k, \rho)$ on M, then
(c)

$$(\mathcal{L}_\nu T)^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} = v^c \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell} - \sum_{i=1}^k T^{a_1 \cdots c \cdots a_k}{}_{b_1 \cdots b_\ell} \nabla_c v^a + \sum_{j=1}^\ell T^{a_1 \cdots a_k}{}_{b_1 \cdots c \cdots b_\ell} \nabla b_j v^c.$$

Proof. We only prove the first part. Given $\mu_a \in \mathcal{T}(0, 1)$ and $w^a \in \mathcal{T}(1, 0)$, we have

$$\begin{aligned}\mathcal{L}_v(\mu_a w^a) = v(\mu_a w^a) &= v^c \nabla_c (\mu_a w^a) \\ &= (v^c \nabla_c \mu_a) w_a + \mu_a v^c \nabla_c w^a.\end{aligned}$$

Since $\mathcal{L}_v$ satisfies Leibnitz rule, we can also write this as

$$\begin{aligned}\mathcal{L}_v(\mu_a w^a) &= \mathcal{L}_v(\mu_a) w^a + \mu_a \mathcal{L}_v(w^a) \\ &= \mathcal{L}_v(\mu_a) w^a + \mu_a \left(v^c \nabla_c w^a - w^c \nabla_c v^a\right).\end{aligned}$$

Equating the two expressions for the Lie derivative gives us the desired relationship. □

5.3 Killing Vector Fields

Definition: A vector field V^c on M is a **kiling vector field** if and only if $\nabla_{[a}v_{b]} = 0$, for the Levi-Civita connection.

Lemma 4: If v^a is a killing vector field, and γ is a geodesic with tangent vector $U^a = \frac{d\gamma(\tau)}{d\tau}$, then

$$\frac{d}{d\tau}(v_a u^a) = 0$$

along γ.

Proof. Note that

$$\begin{aligned}\frac{d}{d\tau}(v^a u^a) &= u^b \nabla_b (v_a u^a) \\ &= u^b u^a \nabla_b v_a + v_a u^b \nabla_b u^a \\ &= 2u^b u^a \nabla_{[b} v_{a]} \\ &= 0.\end{aligned}$$

Note that $u^b u^a$ is symmetric and $\nabla_{[b}v_{a]}$ is antisymmetric, so the above expression is zero. □

We can also count symmetries. On $\mathbb{R}^n$, we have n translations and $\binom{n}{2}$ rotations, for a total of $n + \binom{n}{2}$ symmetries.

If v^c is a Killing field, we have

$$\begin{aligned}R^d_{abc} v_d &= \nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c \\ &= \nabla_a \nabla_b v_c + \nabla_b \nabla_c v_a \\ R^d_{bca} v_d &= \nabla_b \nabla_c v_a + \nabla_c \nabla_a v_b \\ R^d_{cab} v_d &= \nabla_c \nabla_a v_b + \nabla_a \nabla_b v_c.\end{aligned}$$

Adding the second line and subtracting the third, we get

$$(R^d_{abc} + R^a_{bca} - R^d_{cab}) v_d = 2\nabla_b \nabla_c v_a.$$