Wormholes in General Relativity: Just how feasible are they?

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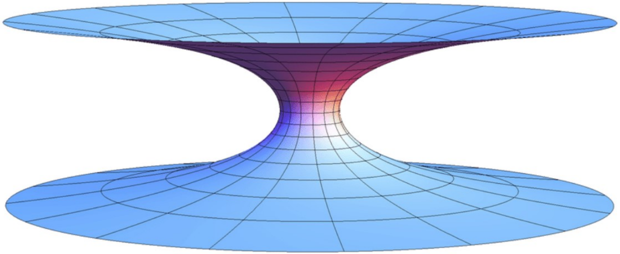
Motivation

- Consider an inter-universe wormhole: allows for maximal symmetry



- Metric given by

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{1}$$

Metric

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - b(r)/r} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right),$$

- Connects the two universes at $r = r_0$
- Universes can be identified by having two coordinate charts $[r_0, \infty)$
- Proper radial distance given by

$$\ell(r) = \pm\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{1 - b(r)/r}} \tag{2}$$

- Define radius of throat to be $r(\ell = 0) = r_0$

Metric

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right),$$

Proposition 1: In some open neighbourhood near the throat (r_0, r_*), the following inequality holds:

$$b'(r) < \frac{b(r)}{r} \tag{3}$$

Curvature

- Stress energy tensor is $T = \mathrm{diag}(\rho, p_r, p, p)$. Einstein's equation gives at the throat,

$$8\pi\rho = \frac{b'(r_0)}{r_0^2} \tag{4}$$

$$8\pi p_r = -\frac{1}{r_0^2} \tag{5}$$

$$8\pi p = \frac{1}{2r_0^2}(1 + r_0\Phi'(r_0))(1 + b'(r_0)). \tag{6}$$

- Eqaution 4 gives

$$b(r) = b(r_0) + 2\int_{r_0}^{r} 4\pi\rho r^2 \, \mathrm{d}r \equiv 2m(r) \tag{7}$$

Energy Conditions

Proposition 2: At the throat of the wormhole, the following inequality holds:

$$\rho + p \leq 0. \tag{8}$$

- Violates the null energy and weak energy condition,

$$\rho + p_i \geq 0. \tag{9}$$

- Turns out *all* energy conditions are broken. The average null energy condition gives over any null curve Γ,

$$I_\Gamma = \int_\Gamma (\rho - \tau)\xi^2 \,\mathrm{d}\lambda = -\frac{1}{4\pi}\int_{r_0}^{\infty} \frac{1}{r^2} e^{-\Phi(r)} \sqrt{1 - \frac{b(r)}{r}} \,\mathrm{d}r \geq 0. \tag{10}$$

- Only examples of energy conditions being broken are in QFT. Thus, modern wormholes are typically a *quantum gravity* problem.

Traversability

Requirements:

- Traversable in a short amount of time.
- Weak tidal forces.

Tidal forces are the main problem, and we get the conditions:

$$|\Phi'| \leq \frac{2gr_0}{(1 - b'(r))L}$$
$$v^2 \leq \frac{2gr_0^2}{(1 - b'(r))L}$$

Constant Redshift Solution

Consider the case where $\Phi(r) = \Phi_0$.

- Numerous possible shape functions, i.e.

$$b(r) = \sqrt{b_0 r} \tag{11}$$

- This gives

$$\rho + p_r = -2p \tag{12}$$

  which is satisfied globally.

(Almost) Schwarzschild

Consider the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r_0}{r} + \frac{\epsilon}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r_0}{r}} + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right] \tag{13}$$

We can bound

$$I_\Gamma > -\frac{1}{4\pi r_0} \tag{14}$$

[Extra] Neglected History

- Universe is a 3-sphere embedded in $\mathbb{R}^4$. Spatial part of metric is (taking a $\theta = \frac{\pi}{2}$ slice)

$$\mathrm{d}\sigma^2 = R_0^2 \left(\mathrm{d}\psi^2 + \sin^2\psi \, \mathrm{d}\varphi^2\right) \tag{15}$$

- This surface is generated by a curve. Let z point in direction of axis of rotation, and R point in the direction perpendicular to z, in the same plane as circle.

$$\frac{dz}{dr} = \tan\psi \tag{16}$$

- For a Schwarzschild metric, spatial part is

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2 \, \mathrm{d}\varphi^2 \,. \tag{17}$$

- Consider the substitution

$$\cos^2\psi = 1 - \frac{2M}{r} \tag{18}$$

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$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2\,\mathrm{d}\varphi^2\,. \tag{21}$$

- Consider the substitution

$$\cos^2\psi = 1 - \frac{2M}{r} \tag{22}$$

[Extra] Neglected History

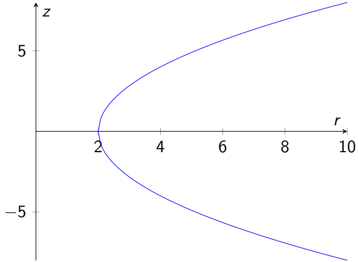
Solving the differential equation

$$\frac{d\psi}{dr} = \sqrt{\frac{2M}{r-2M}} \tag{23}$$

gives

$$z^2 = 8M(r-2M), \tag{24}$$

which is plotted in the figure below, for the case $M = 1$.



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- This surface is generated by a curve. Let z point in direction of axis of rotation, and R point in the direction perpendicular to z, in the same plane as circle.

$$\frac{dz}{dr} = \tan\psi \tag{26}$$

- For a Schwarzschild metric, spatial part is

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2 \, \mathrm{d}\varphi^2 \,. \tag{27}$$

- Consider the substitution

$$\cos^2\psi = 1 - \frac{2M}{r} \tag{28}$$

References

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