CHE374 Final Cheatsheet

Author: QiLin Xue

Interest

The compound interest rate is given by

$$F = P\left(1+\frac{r}{m}\right)^m = P(1+r_{\text{eff}})$$

where r is the nominal interest rate for 1 period (usually for 1 year), and m is the number of times compounded per period.

Equivalence Factors

- $(F/P, i, N) = (1+i)^N$
- $(P/A, i, N) = \frac{(i+1)^N-1}{i(1+i)^N}$
- $(P/G, i, N) = \frac{1}{i^2}\left(1-\frac{1+iN}{(1+i)^N}\right)$
- $(P/Geom, i, g, N) = \frac{1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^N\right)$

For geometric,

$$P = A(P/Geom, i, g, N)$$

For linear,

$$P = G(P/G, i, N).$$

Mortgage

- **Principal:** The amount of money you borrow to pay for a real property
- **Mortgage rate:** the interest rate charged on the mortgage. Compounding period usually matches frequency of payments
- **Amortization Period:** Time horizon for mortgage payment
- **Term:** Duration of time when the mortgage rate is fixed. When term ends, re-evaluate how much you still owe, then use new interest rate to calculate monthly payment based on time left in amortization period.

The net amount owed at end of term is

$$Net = P\left(F/P, i/N, tN\right) - A\left(F/A, i/N, tN\right),$$

where t is the number of years in term and N is the number of payment periods per year. The net monthly payment is

$$A = P\left(A/P, i/N, tN\right),$$

where t is the number of years in amortization (or what is left), and P is the mortgage principal (or what is left).

Bonds

- **Bond:** a type of loan where the creditor pays a stated amount at specified intervals for a defined period (coupon payments), plus a final amount at a specified date (face value)
- **Coupon rate:** the rate used to calculate coupon payments.
- **Coupon payments:** Regular payment made over the course of a bond's lifetime. Amount determined by coupon rate and frequency of payment.

  $$\text{coupon amount} = \text{coupon rate} \times \frac{\text{face value}}{\text{payment frequency}}$$

- **Yield:** Hypothetical interest rate of a bond given a purchase price. Solved using interpolation.

The bond price is given by

$$P = A\left(P/A, i/m, N\right) + F\left(P/F, i/m, N\right),$$

where i is the yield, m is the frequency of coupon payments per time unit, N is the number of periods to maturity ($m \times$ time unit), and A is the value of coupon payment. Unless otherwise stated, assume all coupons are paid semiannually and the payment is half the coupon rate.

Financial Risk

The expected rate of return for a company is given by

$$CAPM = E[R_c] = r_f + \beta_c\left(E[R_{MP} - r_f]\right).$$

Let a be the # of MP shares and b be the # of risk-free shares. Then,

$$aP_{MP+} + bP_{r+} = P_{I+}$$
$$aP_{MP-} + bP_{r-} = P_{I-}$$

where

$$P_I = aP_{MP} + bP_r.$$

Here, the present market price P_{MP} will take on certain values if it goes up or down. The project/investment price P_I will also take on certain values depending on if market goes up or down. The risk-free rate is r_f. We also have,

$$\beta_i = \frac{\sigma_{i,MP}}{\sigma_{MP}^2} = \frac{\rho_{i,MP}\sigma_i}{\sigma_{MP}},$$

where $\sigma_{i,MP}$ is the covariance between ith company and market portfolio, σ_{MP}^2 is the variance of MP, $\rho_{i,MP}$ is the correlation of returns between ith company and MP, σ_i is volatility of ith company, σ_{MP} is volatility of MP. If the fair price at $t=0$ is P_0, then the fair price at time t is

$$P_t = P_0(F/P, CAPM, t).$$

Forward Rates

- Given rates for investments between $t=0$ and $t=t_1$ or $t=t_2$: t_{0,t_1} and t_{0,t_2}, respectively. The interest forward rate t_1 years from $t=0$ for a duration of t_2-t_1 years is

  $$r_{t_1,t_2} = \frac{r_{0,t_2}\cdot t_2 - r_{0,t_1}\cdot t_2}{t_2-t_1}.$$

Example Problems

⋆ Suppose an annual yield rate of 7%, (semi-annual compounding), face value \$100. A bond with a coupon rate of 6% maturing in 3 years and 4 months from now is

$$\left(100\cdot\frac{6\%}{2}(P/A, \frac{7\%}{2}, 7) + 100(P/F, \frac{7\%}{2}, 7)\right)\left(1+\frac{7\%}{2}\right)^{2/6}$$

⋆ CAD risk free rate is 3%. US risk free rate is 3%. Current FX is $0.9US/CAD$, what is the fair FX rate 1 year from now?

- Path 1: Let interest grow on \$1 CAD, gives \$1.03 CAD.
- Path 2: Convert \$1 CAD to \$0.9 USD, let interest grow, gives \$0.9225 USD. Then,

  $$FX_1 = \frac{0.9225}{1.03} = 0.896USD/CAD$$

⋆ Given house price for \$2,995,000, and we have \$599,000 (enough for 20% down payment). Mortgage rate is 2%, with a 5 year term, and a 25 year amortization period. The monthly interest is

$$(1+2\%/2)^2 = (1+i)^{12} \implies i = 0.16598\%.$$

The monthly payment is

$$2,396,000 = A(P/A, i, 300) \implies A = \$10,145.89.$$

⋆ At the end of term, new mortgage rate is 4%. The amount owed is

$$2,396,000 - 10,145.89(P/A, 2\%/12, 60) = 2,007,135.$$

Since there are 20 years left in amortization period, the effective monthly interest is

$$(1+4\%/2)^2 = (1+i)^{12} \implies i = 3.967\%,$$

so the annuity is

$$2,007,135 = A(P/A, i, 240) \implies A = \$12,128.04.$$

$\star$ Consider mortgage P, N total payments at an interest r (based on N compounding periods). If the mth payment ($m < n$) has just been paid, the remaining principal is

$$P_m = \frac{P\left[1 - (A/P, r, N)(P/A, r, m)\right]}{(P/F, r, m)}$$

$\star$ Consider a 5-year mortgage of \$250,000, with a quoted interest rate of 12.304% per year with monthly payments. The monthly rate is

$$(1 + 12.304\%/2)^2 = (1+i)^{12} \implies i = 1\%.$$

$\star$ The principal left on mortgage right after the monthly payment was made at the end of year 2 is \$167,421. The interest charged in the first month of year 3 is given by

$$167,421 \times 1\% = 1674.31.$$

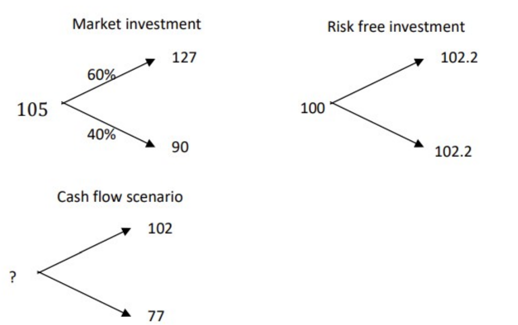
$\star$ What would be the break-even penalty fee that you would be willing to pay if we enter into new mortgage contract at the beginning of year 3, with rate quoted at 6.0755%. We have a break-even when monthly payments are the same,

$$(1 + 6.0755\%/2)^2 = (1+i_2)^{12} \implies i_2 = 0.5\%.$$

Then, we can solve for x_{penalty},

$$5561.11 = (167,431.21 + x_{\text{penalty}})(A/P, 0.5\%, 3 \times 12)$$

$\star$ Current market index price is \$105. In the next time period, the market can either go up to \$127, with probability 60%, or down to \$90. The risk-free rate is 2.2% per time period. You have an opportunity to buy a cash flow scenario where if the market goes up, you will receive \$102.00 and if the market goes down, you will receive \$77. Determine a fair market price for this scenario. What is the beta for this scenario? We have,



,

which gives us

$$\begin{aligned} 127a + 102.2b &= 102 \\ 90a + 102.2b &= 77, \end{aligned}$$

which gives $(a, b) = (0.676, 0.158)$. The fair price would then be $105a + 100b = \$86.79$. The market expected rate of return is $(127 \cdot 60\% + 90 \cdot 40\%)/105 - 1 = 6.857\%$ and cash flow expected rate of return is $(102 \cdot 60\% + 77 \cdot 40\%)/86.79 - 1 = 6.007\%$, so

$$\beta = \frac{6.007\% - 2.2\%}{6.857\% - 2.2\%} = 0.8175.$$

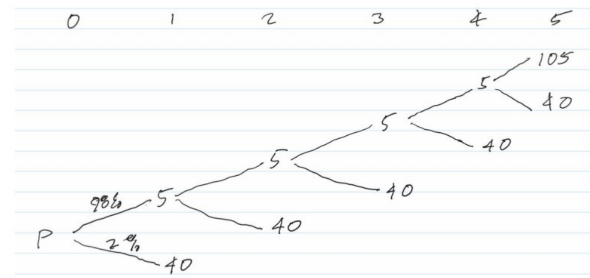
$\star$ In the Canadian market, the market index (portfolio) is currently trading at C\$10. If the market goes up, the price is expected to be C\$12 and if the market goes down, the price is expected to be C\$9. The Canadian risk-free rate is 5% for the period. In the US, the current price of the market index is US\$100. If the market goes up, its price is expected to be US\$110, and if the market goes down, the price is expected to be US\$95. The current exchange rate is 0.80 US/C. Assume the Canadian and US markets are 100% correlated. Determine the US risk-free rate. (Hint: write out the US market payoff in CAD assuming the FX rate one year from now to be FX1 and using no-arbitrage arguments you should be able to solve for FX1, then solve for the interest rate.) We have (no arbitrage),

$$\begin{aligned} 12a + 1.05b &= 110/FX_1 \\ 9a + 1.05b &= 95/FX_1 \\ 10a + b &= 100/0.8, \end{aligned}$$

where we converted the USD graph to CAD (hence why we divide by FX_1 and $FX_0 = 0.8$). This gives $a = 6.40$, $b = 61.0$, and $FX_1 = 0.781$. Then using the standard conversion (let CAD collect interest, or convert CAD to USD, collect interest, and convert back), we get

$$0.8(1 + r_{US}) \times \frac{1}{0.781} = 1.05 \implies r_{US} = 2.5\%.$$

$\star$ Wish to calculate yield rate for a client company, RSG, that is interested in raising \$10,000,000 through 5 year term bonds with annual coupon payments. Risk-neutral probability of going bankrupt, and thus defaulting on making its future payments is 2% per coupon payment term (i.e. per year). Assume that the "loss given default" equals 40% - i.e. if RSG goes bankrupt, no more coupons will be paid and the investors will receive 40% of the face value of the bond. The risk-free rate is 3% (annual, effective). You estimate RSG's weighted average cost of capital to be 7%. The coupon rate is 5%. The expected cash flow is,



where

$$\begin{aligned} P_0 &= (0.98 \cdot P_1 + 0.02 \cdot 40)/1.03 \\ P_1 &= 5 + (0.98 \cdot P_2 + 0.02 \cdot 40)/1.03 \\ &\cdots\cdots \\ P_4 &= 5 + (0.98 \cdot 105 + 0.02 \cdot 40)/1.03 = 105.679, \end{aligned}$$

and work backwards to solve for the price of the bond $P_0 = \$103.083$. The yield is therefore given by

$$103.083 = 100 \cdot 5\%(P/A, Y, 5) + 100(P/F, Y, 5).$$

Note that if the first payment period is $f\%$ of the regular period, we divide by $1.03^{f\%}$, and the probability becomes $p \mapsto fp$.

$\star$ Given a bond yield curve where at year 3 the continuously compounded yield is 4.3% and at year 8 the continuously compounded yield is 5.82%, at what forward rate should an investor lock into for an investment in a bond, 3 years from now, for 5 years. That is, the investor plans to look into a rate to invest in the bonds for 5 years 3years from now (i.e. a forward rate). We have,

$$F_{0,3yr} = P_0 e^{r_{3yr} t_3}; \qquad F_{0,8yr} = P_0 e^{r_{8yr} t_8}.$$

Let $r_{3yr,8yr}$ be the forward rate. Then, the following are true:

$$\begin{aligned} F_{0,8yr} &= F_{0,3yr} e^{r_{3yr,8yr}(t_8 - t_3)} \\ P_0 e^{r_{8yr} t_8} &= P_0 e^{r_{3yr} t_3} e^{r_{3yr,8yr}(t_8 - t_3)} \\ r_{3yr,8yr} &= \frac{r_{8yr} t_8 - r_{3yr} t_3}{t_8 - t_3} = 6.732\%. \end{aligned}$$

For example, we can compute,

$$F_{3,8} = F_3 e^{r_{3,8} \times 5},$$

where $F_{3,8}$ is the future value at $t = 8$ given present at $t = 3$.