ECE286: Stats

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1 Cumulative Distribution Functions

Given some PDF $f(x)$, the **cumulative distribution function** (CDF) is defined as:

$$CDF(X) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t \tag{1.1}$$

where $CFD(\infty) = 1$. Recall that given some function $g(x)$, we have

$$\int_A^C g(x)\,\mathrm{d}x = \int_A^B g(x)\,\mathrm{d}x + \int_B^C g(x)\,\mathrm{d}x\,. \tag{1.2}$$

Therefore, if we have some random value x which has a PDF $f(x)$. Then:

$$P(A \le X \le B) = F(B) - F(A). \tag{1.3}$$

Let us now determine the **normal CDF.** Recall that

$$n(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{1.4}$$

And therefore the CDF is given by

$$\Phi(x, \mu, \sigma) = \int_{-\infty}^{x} n(t; \mu, \sigma)\,\mathrm{d}t\,, \tag{1.5}$$

so $P(A \le X \le B, \mu, \sigma) = \Phi(B, \mu, \sigma) - \Phi(a, \mu, \sigma)$. Note that Φ cannot be written in terms of elementary functions. In practice, we will use tables to evaluate this. However, we don't want tables for every μ and σ, so we will parametrize it by a single variable and relate it to the **CDF for a standard normal RV**,

$$\Phi(x) = \int_{-\infty}^{t} n(t; 0, 1)\,\mathrm{d}t\,. \tag{1.6}$$

Suppose X has PDF $n(x; \mu, \sigma)$. Let $z = \dfrac{x-\mu}{\sigma}$. Consider:

$$\begin{aligned}
P(X \le x) &= \int_{-\infty}^{x} n(t; 0, 1)\,\mathrm{d}t && (1.7)\\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t && (1.8)\\
&= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-s^2/2}\,\mathrm{d}t && (1.9)\\
&= \int_{-\infty}^{(x-\mu)/\sigma} n(s; 0, 1)\,\mathrm{d}s && (1.10)\\
&= P\left(\frac{x-\mu}{\sigma}\right) && (1.11)\\
&= P\left(Z \le \frac{x-\mu}{\sigma}\right). && (1.12)
\end{aligned}$$

Note: In MATLAB, the code for Φ is `normcdf`.

2 Binomial PMF

Recall that when we have n coin flips, p is the probability of heads, and X is the number of heats. Recall that the **probability mass function** is

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}. \tag{2.1}$$

The mean is np and the variance is $np(1-p)$. Then:

$$z = \frac{x - np}{\sqrt{np(1-p)}}. \tag{2.2}$$

A preview of the **central limit theorem** is that in the limiting case of $n \to \infty$, the distribution of X is the normal distribution.

3 Gamma Function

The **gamma function** is defined as:

$$\Gamma(z) = \int_0^\infty x^{\alpha-1} e^{-x} \, \mathrm{d}x \tag{3.1}$$

for $\alpha > 0$. IT has the following properties:

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- $\Gamma(n) = (n-1)!$ where $n \in \mathbb{N}$.

The **gamma distribution** is

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

The mean is $\mu = \alpha\beta$ and the variance is $\sigma = \alpha\beta^2$.

The gamma distribution is special since many other distributions can be written as it. For example, the **chi-squared** distribution χ^2 is

$$f(x; v) = f_{\text{gamma}}(x; v/2, 2) = \begin{cases} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{3.3}$$

The **exponential distribution is**

$$f(x; \beta) = f_{\text{gamma}}(x; 1, \beta) \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{3.4}$$

Let us relate this to the poisson PMF,

$$p(x; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}. \tag{3.5}$$

We can interpret this by setting $\lambda = rt$, where r is the rate of arrivals and t is the length of the interval. Therefore, the probability of no arrivals in an interval of length t is:

$$p(0; rt) = e^{-rt}. \tag{3.6}$$

Let Y be a RV for the time to first arrival. Then:

$$P(Y > t) = e^{-rt}. \tag{3.7}$$

Therefore,

$$P(Y \leq t) = 1 - e^{-rt} = F(t), \tag{3.8}$$

so the PDF of Y is

$$f(t) = \frac{d}{dt} F(t) = re^{-rt}. \tag{3.9}$$

4 Memoryless Property

Suppose X is exponential with $\beta > 0$. Then:

$$P(X > s + t | X > s) = \frac{P(X > s + t \cap X > s)}{P(X > s)} = \frac{P(X > s + t)}{P(x > s)}. \tag{4.1}$$

This is equal to

$$= \frac{\int_{s+t}^\infty \frac{1}{\beta} e^{-x/\beta} \, \mathrm{d}x}{\int_s^\infty \frac{1}{\beta} e^{-x/\beta} \, \mathrm{d}x} \tag{4.2}$$

$$= \frac{e^{-(s+t)/\beta}}{e^{-s/\beta}} \tag{4.3}$$

$$= e^{-t/\beta} \tag{4.4}$$

$$= P(X > 1). \tag{4.5}$$