MAT357: Real Analysis
Problem Set 2

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I discussed problems Ahmad, Jonah, Andy, and Nathan for this problem set (different for each question).

1. (a) Let (a_n) be a converging sequence in M that converges to a. We wish to show that any isometry f sends this converging sequence to another converging sequence in N. That is, we wish to show that $(f(a_n))$ converges to $f(a)$ in N. First, since (a_n) is converging, we know that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that

$$d_M(a_n, a) < \epsilon. \tag{0.1}$$

Now working with the sequence $(f(a_n))$ in N, for all $\epsilon > 0$, we can pick the same N as before. Then,

$$d_M(a_n, a) < \epsilon \implies d_N(f(a_n), f(a)) < \epsilon, \tag{0.2}$$

where the implication is given by the fact that f is an isometry.

(b) A homeomorphism is a bijective continuous function with a continuous inverse. By definition, it is continuous, and we have shown that it is continuous. It remains to show that its inverse is also continuous.

Let $g := f^{-1}$. We wish to show that g is also an isometry, or equivalently for any $p, q \in N$ we have

$$d_N(p, q) = d_M(g(p), g(q)) \tag{0.3}$$

This is true since if f is an isometry, we can find $p', q' \in M$ such that $f(p') = p$ and $f(q') = q$. Then by definition, the following chain of implications hold:

$$d_N(f(p'), f(q')) = d_M(p, q) \tag{0.4}$$
$$\implies d_N(f(p'), f(q')) = d_M((f^{-1} \circ f)p, (f^{-1} \circ f)q) \tag{0.5}$$
$$\implies d_N(p, q) = d_M(g(p), g(q)). \tag{0.6}$$

This is true for all p, q so f^{-1} is an isometry. But we've shown that isometries are continuous, so f^{-1} is continuous, and we are finished.

(c) Suppose for the sake of contradiction that $[0, 1]$ is isometric to $[0, 2]$. We will make use of the following lemma:

> **Lemma 1:** Let $f : M \to N$ be a homeomorphism between two compact sets in $\mathbb{R}^n$. If $m \in \partial M$ then $f(m) \in \partial N$.
>
> *Proof.* Suppose for contradiction that the above is not true. That is, there exists a homeomorphism $f : M \to N$ and $m \in \partial M$ such that $f(m) \in \text{int}(N)$. If this was true, then we can consider an open ball $B_\delta \in \text{int}(N)$ around $f(m)$ for some $\delta > 0$. Then the preimage of this open ball will be an open ball in $\text{int}(M)$ that contains m. This contradicts our assumption that m is on the boundary, and thus not in the interior of M. □

Let $f : [0, 1] \to [0, 2]$ be an isometry. Then by the above lemma, then we either have $f(0) = 0, f(1) = 2$ or $f(0) = 2, f(1) = 0$. In either case,

$$d_{[0,1]}(0, 1) = 1 \tag{0.7}$$

and

$$d_{[0,2]}(f(0), f(1)) = d_{[0,2]}(0, 2) = d_{[0,2]}(2, 0) = 2. \tag{0.8}$$

Since $1 \neq 2$, we have contradicted the assumption that f is an isometry.

2. We first make use of the following lemma,

> **Lemma 2:** Every Cauchy sequence (a_n) is bounded.
>
> *Proof.* We wish to show that there is some M such that $|a_n| < M$. Because (a_n) is Cauchy, we have that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n_1, n_2 \geq N$ implies that
>
> $$d(a_{n_1}, a_{n_2}) < \epsilon. \tag{0.9}$$
>
> Suppose that the sequence is unbounded. Then we can choose $\epsilon = 1$. For some $N \in \mathbb{N}$ we have for all $n_1, n_2 > N$,
>
> $$d(a_{n_1}, a_{n_2}) < 1 \tag{0.10}$$
>
> But by the reverse triangle inequality, we have that
>
> $$d(a_{n_1}, a_{n_2}) \geq d(|a_{n_1}|, |a_{n_2}|). \tag{0.11}$$
>
> Because (a_n) is unbounded, there exists $n_1 > N$ such that $|a_{n_1}| > 2 + |a_{n_2}|$. This implies that
>
> $$d(a_{n_1}, a_{n_2}) \geq 2. \tag{0.12}$$
>
> But this contradicts the statement that $|a_{n_1} - a_{n_2}| < 1$, so we have a contradiction and (a_n) has to be bounded. □

Consider a Cauchy sequence (a_n) contain in M. The above lemma implies that there exists a closed set S such that $a_n \in S$ for all $n \in \mathbb{N}$. This set is bounded, and by the assumption given in the problem, S is compact.

By definition, because S is compact, every sequence (b_n) has a convergent subsequence (b_{n_k}). If (b_n) converges in S, then (b_{n_k}) must converge to the same limit point in S.

Every Cauchy sequence has a limit point. What remains to be shown is that the limit point of (a_n) is contained in S. But because S is compact, we know that there is a subsequence that converges to a point in S. Since (a_n) converges to its limit point, this limit point must be the same as the limit point of its converging subsequence, which is contained in S.

3. (a) Suppose the graph is not closed. Then there exists a sequence $y_n := (a_n, f(a_n))$ such that its limit point, which we denote as $y := (a, b)$, is not contained in the graph. This means that $b \neq f(a_n)$ (because if they were equal, then $y = (a, f(a))$ would be contained in the graph).

Because $(a_n, f(a_n))$ has a limit point, then so does the component sequence (a_n) and $(f(a_n))$ which converges to a and b respectively.

Consider the sequence $(a_n) \subset M$, which converges to $a \in M$. Then because f is continuous, we must have that $(f(a_n))$ converges to $f(a) \in \mathbb{R}$, so the limit point of $(a_n, f(a_n))$ is $(a, f(a))$. But we already said that the limit point was actually (a, b) with $b \neq f(a)$, leading to a contradiction.

(b) If f is continuous and M is compact, then $f(M)$ is compact. This is true since we can take any sequence (a_n) in M and find a convergent subsequence (a_{n_k}) that converges to a. Then by continuity, we have that for the sequence $f(a_n) \in \mathbb{R}$, there exists a subsequence $f(a_{n_k})$ that converges in $\mathbb{R}$ to $f(a)$, so $f(M)$ is compact.

Consider any sequence $y_n := (a_n, f(a_n))$ on the graph. We know that (a_n) has a converging subsequence and we just proved that because $f(M)$ is compact, $(f(a_n))$ has a converging subsequence. Assume these subsequences are given by $(a_{n_k}), (f(a_{n_k}))$, Therefore, $(a_n, f(a_n))$ has the converging subsequence $(a_{n_k}, f(a_{n_k}))$ which converges to the point $(a, b) \in M \times \mathbb{R}$.

(c) Consider a converging sequence (a_n) that converges to $a \in M$. We wish to show that $(f(a_n))$ converges in $\mathbb{R}$. First, consider the sequence $((a_n, f(a_n)))$. Because the graph is compact, this has a converging subsequence $(a_{n_k}, f(a_{n_k}))$ that converges to a point (a, b) on the graph. Note that (a_{n_k}) converges to a since a convergent subsequence of a convergent sequence has the same limit point. But every point on the graph can be written as $(x, f(x))$, we must have $b = f(a)$. So we have that $(f(a_{k_n}))$ converges to $f(a)$.

We have shown that any converging subsequence $(f(a_{n_k}))$ converges to $(f(a))$. We then consider the following lemma.

> **Lemma 3:** Let $(x_n) \subset M$ be a sequence, where $M \subset \mathbb{R}$ is a compact set. If all converging subsequences have the same limit point x, then (x_n) converges to x.
>
> *Proof.* Suppose for the sake of contradiction that (x_n) does not converge to x. Then there exists some $\epsilon > 0$ such that for any choice of $N \in \mathbb{N}$, there exists some $n > N$ such that
>
> $$d(x_n, x) \geq \epsilon. \tag{0.13}$$
>
> Let X be the sequence of all x_n such that $d(x_n, x) \geq \epsilon$ with $n > N$. This is an infinite set because if it was finite, then we could pick a higher N value such that there are no $n > N$ such that $|x_n - x| \geq \epsilon$.
> Now, we can treat $X = (a_n)$ as a sequence. Because M is compact, there is a converging subsequence (a_{n_k}) that converges to x. This means that for all $\epsilon > 0$ (including the one we chose above), there exists some $N \in \mathbb{N}$ such that for every $n > N$ we have
>
> $$d(a_{n_k}, x) < \epsilon. \tag{0.14}$$
>
> But we defined a_{n_k} to be in the set of the of x_n such that $d(x_n, x) \geq \epsilon$, leading to a contradiction. □

The properties of the above lemma hold, so $(f(a_n))$ converges. Therefore, f is continuous since it maps convergent sequences in M to convergent sequences in $\mathbb{R}$.

(d) Consider the function

$$f(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{0.15}$$

Then this function is clearly not continuous, since we can consider the sequence $(a_n) = -\dfrac{1}{1}, -\dfrac{1}{2}, -\dfrac{1}{3}, \ldots$. This clearly converges to 0 but $(f(a_n))$ does not converge since $(f(a_n)) = (f(1/n)) = (n)$, which is unbounded. However, this graph is closed since it contains all its limit points. Consider a converging sequence $p_n := (a_n, f(a_n))$. There are three cases:

- The sequence p_n contains an infinite number of points $(0, 0)$. If this occurs, then it converges to $(0, 0)$ which is contained in the graph.

- The sequence p_n contains an infinite number of points $(a_n, f(a_n))$ where $a_n > 0$. If this occurs, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $a_n > 0$ so we are restricted to the graph $\{(p, y) \in (0, \infty) \times \mathbb{R} : y = 1/p\}$. This is continuous, so the graph is closed (by part (a)), so the limit point of p_n is contained in the graph.
- The same argument as the previous case, but with an infinite number of points where $a_n < 0$.

4. We are given that $K_1 \supset K_2 \supset \cdots$ and $\text{diam}(K_i) \geq \mu$. Let us define

$$\mu' = \sup\{\mu : \text{diam}(K_i) \geq \mu\}. \tag{0.16}$$

> **Lemma** 4: The sequence $(\text{Diam}(K_n))$ converges to μ'.
>
> *Proof.* Because $K_n \supset K_{n+1}$, then $(\text{Diam}(K_n))$ is a non-increasing function that is bounded below by 0. Therefore, a limit definitely exists, and specifically the limit point is the infimum of the sequence, which is by definition μ'. □

Next, we can show that $\mu' \in \{\mu : \text{diam}(K_i) \geq \mu\}$. This is true because $(\text{diam}(K_i))$ is a non-increasing converging sequence, so this is a closed set, and the supremum is contained in the set.

Note that

$$\text{diam}(K) = \text{diam}\left(\bigcap K_i\right) = \inf\{\text{diam}(K_i)\} = \mu'. \tag{0.17}$$

The last equality is true since the infimum is the largest such μ such that $\text{diam}(K_i) \geq \mu$, which we defined as μ'. Therefore, since $\mu' \geq \mu$ we have $\text{diam}(K_i) \geq \mu$.

5. (a) We will prove this for a general complete metric space N, and since $\mathbb{R}$ is a complete metric space, then we are done. We will do this in a few steps:

(1) Consider a sequence $(a_n) \subset S$ that converges to $Lx \in \partial S$. Given this, we will show that $(f(a_n))$ converges to some Y.

(2) Consider a different sequence $(b_n) \subset S$ that converges to the same limit point as above, x. We will show that $(f(b_n))$ converges to the same point as above, Y.

(3) We extend f to $\bar{f}$ by defining $\bar{f}(x) = f(x)$ for all $x \in S$ and $\bar{f}(x) = \lim_{n\to\infty} a_n$ for any (a_n) that converges to $x \in \bar{S} \setminus S$. This is a well-defined function by the above, and it remains to show that this is uniformly continuous.

We will do the above:

(1) The sequence (a_n) converges, so it is also a Cauchy sequence. Because f is uniformly continuous, for every $\epsilon > 0$ there exists a $\delta > 0$ such that given $a_i, a_j \in S$, we have $d(a_i, a_j) < \delta \implies d(f(a_i), f(a_j)) < \epsilon$. Because (a_n) converges, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $i, j > N$ implies that $d(a_i, a_j) < \epsilon$. If we set $\epsilon = \delta$, then we've shown that

$$i, j > N \implies d(a_i, a_j) < \delta \implies d(f(a_i), f(a_j)) < \delta,$$

so $(f(a_n))$ is Cauchy. But because N is complete, this Cauchy sequence must converge. Let us denote this limit point to be Y.

(2) We wish to show that given any sequence (b_n) that also converges to x, the sequence $(f(b_n))$ also converges to Y. To do this, we need to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N \implies d(Y, f(b_n)) < \epsilon$. To show this, we make use of the triangle inequality:

$$d(Y, f(b_n)) \le d(Y, f(a_n)) + d(f(a_n), f(b_n)). \tag{0.18}$$

Both terms can have an arbitrary upper bound. Because $(f(a_n))$ converges to Y there exists $N_a \in \mathbb{N}$ such that $d(Y, f(a_n)) < \frac{\epsilon}{3}$. Note that for every $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $n > N$ implies that $d(a_n, b_n) < delta$. This is true since by the triangle inequality,

$$d(a_n, b_n) \le d(a_n, x) + d(x, b_n) < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{0.19}$$

which is true since both (a_n) and (b_n) converge, so they can get arbitrarily close to x.

By uniform continuity there exists $\delta > 0$ such that $d(a_n, b_n) < \delta \implies d(f(a_n), f(b_n)) < \frac{\epsilon}{3}$. We showed that for any choice of δ we can pick n such that $d(a_n, b_n) < \delta$, so we are able to bound the second term in the original triangle inequality by $\frac{\epsilon}{3}$ as well. We have, for all $\epsilon > 0$, a choice of $N \in \mathbb{N}$ such that $n > N$ implies that

$$d(Y, f(b_n)) \le d(Y, f(a_n)) + d(f(a_n), f(b_n)) < \frac{2\epsilon}{3} < \epsilon \tag{0.20}$$

so $(f(b_n))$ converges to the same Y.

(3) Finally, we extend f to $\bar{f}$ by defining $\bar{f}(x) = f(x)$ for all $x \in S$ and $\bar{f}(x) = \lim_{n\to\infty} a_n$ for any sequence (a_n) that converges to $x \in \bar{S} \setminus S$. We wish to show that for any $x, y \in \bar{S}$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta \implies d(\bar{f}(x), \bar{f}(y)) < \delta$. There are three cases to consider here:

- If $x, y \in S$, then $\bar{f} = f$ and since f is uniformly continuous we are done.
- Let $x \in S$ and $y \in \bar{S} \setminus S$. Consider a sequence (a_n) that converges to y.

  Because the sequence converges, for any $N \in \mathbb{N}$ there exists some $\delta > 0$ such that $d(a_n, y) < \delta \implies n > N$. Then for every $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $n > N$ implies that $d(f(a_n), \bar{f}(y)) < \epsilon$. Therefore, we have shown that $d(a_n, y) < \delta \implies d(f(a_n), \bar{f}(y)) < \epsilon$. Since x can be part of a sequence that converges to y, then we're done.
- Let $x, y \in \bar{S} \setminus S$. Then, we use the triangle inequality. Consider,

$$d(x, y) \le d(x, z) + d(z, y) \tag{0.21}$$

where $z \in S$. We have shown in the previous case that for all $\epsilon_x, \epsilon_y > 0$, there exists $\delta_x, \delta_y > 0$ such that $d(x,z) < \delta_x \implies d(\bar{f}(x), f(z)) < \epsilon_x$ and $d(z,y) < \delta_y \implies d(f(z), \bar{f}(y)) < \epsilon_y$. If we choose $\epsilon_x, \epsilon_y = \frac{\epsilon}{3}$ and $\delta' = \min\{\delta_x, \delta_y\}$, then it follows that $d(x,z), d(z,y) < \delta'$ implies that $d(\bar{f}(x), f(z)) + d(f(z), \bar{f}(y)) < \frac{2\epsilon}{3} < \epsilon$.

To finish our proof that $\bar{f}$ is continuous, consider any $\epsilon > 0$ and some arbitrary $z \in S$. Then there exists a $\delta = 2\delta'$ where δ' is the value from the previous paragraph which corresponds to $\epsilon_x, \epsilon_y = \frac{\epsilon}{3}$ such that

$$d(x,y) < \delta. \tag{0.22}$$

By how δ is defined, we must also have:

$$d(x,z) + d(z,y) < \delta. \tag{0.23}$$

But we have shown that this implies

$$d(\bar{f}(x), f(z)) + d(f(z), \bar{f}(y)) < \epsilon.$$

Therefore, $d(x,y) < \beta$ implies that,

$$d(\bar{f}x, \bar{f}y) < d(\bar{f}(x), f(z)) + d(f(z), \bar{f}(y)) < \epsilon$$

and we are done.

Because $\mathbb{R}$ is a complete metric space, then the above holds.

(b) We wish to show that this extension is unique. Consider a function $g \neq \bar{f}$ that also extends f that is uniformly continuous. This means that there exists a point $p \in \bar{S} \setminus S$ such that $g(p) \neq \bar{f}(p)$. Then we claim that the above is impossible, i.e. there is a contradiction.

Consider a sequence $(a_n) \subset S$ that converges to p. Then for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that,

$$d(a_n, p) < \delta$$

and because g is uniformly continuous, it means that for very $\epsilon > 0$ and $x, y \in \bar{S}$, there exists $\delta' > 0$ such that

$$d(x,y) < \delta' \implies d(g(x), g(y)) < \epsilon$$

Let $x = a_n$ and $y = p$. Then

$$\begin{aligned} n > N &\implies d(a_n, p) < \delta \\ &\implies d(a_n, p) < \delta' \\ &\implies d(f(a_n), g(p)) < \epsilon. \end{aligned}$$

On the second line, we were able to set $\delta = \delta'$ since we can find a $N \in \mathbb{N}$ for any δ value. For the third line, we used the fact that f agrees with g in S. We have shown that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(f(a_n), g(p)) < \epsilon,$$

or equivalently, $(f(a_n))$ converges to $g(p)$. But we have shown in the previous part that $(f(a_n))$ converges to $\bar{f}(p)$, so we have

$$g(p) = \bar{f}(p),$$

which violates our assumption that they do not agree at this point.

(c) See above.

6. (a) Suppose that M is not compact. Then there exists a sequence (a_n) with no convergent subsequence, so the sequence as a set is closed. If it was not closed then there would be a subsequence that that converges to a point not in this set, so M would not be compact. Therefore, the distance between any two distinct points will be nonzero. We then define:

$$\delta_i = \frac{1}{3}\inf\{d(a_i, a_j) : a_i \neq a_j\}. \quad (0.24)$$

Then consider the closed ball around point a_i with radius δ_i, given by:

$$B_i(x) = \begin{cases} 1 & \text{if } d(x, a_i) \leq \delta_i \\ 0 & \text{otherwise} \end{cases} \quad (0.25)$$

as well as the function,

$$f(x) = \sum_{i=1}^{\infty} (-1)^i i B_i(x)(1 - d(x, a_i)). \quad (0.26)$$

We wish to show that this is continuous. We look at different cases:

- The function is continuous outside the closed balls, because then we have $f = 0$, which is continuous.
- Inside the closed balls, only one of the terms in the sum is nonzero. So we need to show that

$$(-1)^i i B_i(x)(1 - d(x, a_i))$$

  is continuous. Here, $B_i(x) = 1$ and i is a constant. The distance function is continuous, so a linear combination of this distance function must also be continuous.
- Finally, we need to show the function is continuous at the boundary of the support of this function. A sequence (b_n) that approaches X in this boundary from outside the balls will have the corresponding sequence $(f(a_n)) = 0, 0, \ldots,$ which converges to 0. A sequence (c_n) that approaches the same X in this boundary from inside the balls will have the corresponding sequence $(f(c_n))$ which converges to $f(X) = 0$. These agree, so f must be continuous.

However, this function is not bounded above because for any $M > 0 \in \mathbb{R}$ we know that

$$f(a_{2\lceil M \rceil + 4}) = 2\lceil M \rceil + 4 > M.$$

Similarly, it doesn't have a lower bound since for any $M < 0 \in \mathbb{R}$ we know that

$$f(a_{2\lfloor M \rfloor - 3}) = 2\lfloor M \rfloor - 3 < M.$$

So we proved the contrapositive.

(b) This is a similar question. We again prove the contrapositive. Consider M is not compact. We can perform the same construction to arrive at the function

$$f(x) = \sum_{i=1}^{\infty} (-1)^i i B_i(x)(1 - d(x, a_i)).$$

Now consider the function.

$$g(x) = \tanh(f(x))$$

This is continuous because $\tanh$ is continuous and f is continuous, and the composition of continuous functions is continuous. It is also bounded since the codomain of $\tanh$ is $(-1, 1)$. However, it doesn't reach a maximum or a minimum since $f(x)$ is unbounded on both sides.