MAT357: Real Analysis
Problem Set 5

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I worked with Ahmad, and Nathan on this problem set.

1. We will make use of the following lemma,

> **Lemma 1:** If f is continuous and constant on the rationals, then it is constant on the reals.
>
> *Proof.* Let $f(q) = C$ for all rationals $q \in \mathbb{Q}$. Suppose for the sake of contradiction that f is not constant on the irrationals. That is, there exists some $a \in \mathbb{Q}^c$ such that $f(a) \neq C$. Set $\epsilon = |f(a) - C|$.
>
> Because f is continuous at a, there exists $\delta > 0$ such that
>
> $$|x - a| < \delta \implies |f(x) - f(a)| < \frac{\epsilon}{2}. \tag{0.1}$$
>
> But because the rationals are dense in the reals, there exists some $q \in \mathbb{Q}$ such that $|q - a| < \delta$, and by definition of continuity, we have
>
> $$|f(q) - f(a)| < \epsilon/2. \tag{0.2}$$
>
> But we defined $\epsilon = |f(q) - f(a)|$, so we have a contradiction. Note that we don't actually need the $\epsilon/2$ part and can keep it at ϵ, but the factor of 2 was introduced for emphasis that it's not something weird happening at the boundary of the inequality. ☐

I claim that f is constant on the reals by showing it is constant on the rationals. Suppose for the sake of contradiction that f is not constant on the rationals. Then there exists a rational number $p/q \in \mathbb{Q}$ where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ such that $f(p/q) \neq f(0)$. Let $\epsilon = |f(p/q) - f(0)|$.

Because $f_n(x)$ is equicontinuous, there exists $\delta > 0$ such that $|t| < \delta$ implies that $|f_n(0) - f_n(t)| < \frac{\epsilon}{2}$ for all n. Using the definition of f_n, this implies that

$$|f(0) - f(nt)| < \frac{\epsilon}{2} \tag{0.3}$$

for all n. However, note that there exists $N \in \mathbb{N}$ such that $\left|\frac{p}{Nq}\right| < \delta$. Set $t = \frac{p}{Nq}$ such that we now have:

$$\left|f(0) - f\left(n\frac{p}{Nq}\right)\right| < \frac{\epsilon}{2}. \tag{0.4}$$

for all $n \in \mathbb{N}$. Pick $n = N$ so we now have:

$$\left|f(0) - f\left(\frac{p}{q}\right)\right| < \frac{\epsilon}{2} \implies \epsilon < \frac{\epsilon}{2}, \tag{0.5}$$

a contradiction. Using the lemma, because it is constant on the rationals, and f is continuous, it must be constant on the irrationals as well.

2. (a) WLOG let us set $[a, b] = [0, 1]$ (we can apply a simple rescaling argument later).

Because (f_n) is equicontinuous, pick any $\epsilon > 0$. Then there exists $\delta > 0$ such that $|s - t| < \delta, n \in \mathbb{N}$ implies that $|f_n(t) - f_n(s)| < \epsilon$. There exists $N \in \mathbb{N}$ such that $1/N < \delta$. Then,

$$d(f_n(0), f_n(1)) < d(f_n(0), f_n(1/N)) + d(f_n(1/N), f_n(2/N)) + \cdots + d(f_n((N-1)/N), f_n(1)) \tag{0.6}$$
$$< N\epsilon, \tag{0.7}$$

which is true for all n. This is just a rigorous way of saying that $[0, 1]$ is covered by a finite number of open intervals with radius smaller than δ and the maximum change in f_n in each of those open intervals is smaller than δ, so $d(f_n(0), f_n(1)) < N\epsilon$ for some $N \in \mathbb{N}$.

Also note that the maximum and minimum of f_n is bounded by this same $N\epsilon$, using the same line of reasoning. Because the minimum/maximum exist since f_n is a continuous function on a compact set, we can let $f_n(x_m)$ be a minimum and $f_n(x_M)$ be a maximum.

Because $0 \leq x_m, x_M \leq 1$, the interval $[x_m, x_M]$ is contained in a finite union of the above intervals of radius δ and by the same logic, we have

$$d(f_n(x_m), f_n(x_M)) < N\epsilon. \tag{0.8}$$

This means that each f_n is individually bounded by

$$f_n(p) - N\epsilon < f_n(x) < f_n(p) + N\epsilon \tag{0.9}$$

for some $p \in [0, 1]$. Now because $(f_n(p))$ is bounded, there exists a finite

$$m := \inf\{f_n(p)\} \tag{0.10}$$
$$M := \sup\{f_n(p)\}. \tag{0.11}$$

Then $f_n(x)$ is uniformly bounded since

$$\sup\{f_n(x)\} < \sup\{f_n(p) + N\epsilon\} \leq M + N\epsilon \tag{0.12}$$
$$\inf\{f_n(x)\} > \inf\{f_n(p) - N\epsilon\} \geq m - N\epsilon, \tag{0.13}$$

so

$$m - N\epsilon \leq f_n(x) \leq M + N\epsilon. \tag{0.14}$$

(b) If (f_n) is an equicontinuous sequence of functions in $C^0([a, b], \mathbb{R})$ such that $(f_n(p))$ is bounded, then (f_n) has a uniformly convergent subsequence.

(c) i. For (a, b) : In the previous part, we used the fact that $[a, b]$ is compact to show that each f_n is bounded, but we actually didn't need to do so. Because each f_n is uniformly continuous on a bounded interval, we can create a finite open cover for $(0, 1)$ with intervals of radius δ. In each of these intervals, the function f_n can change by a maximum of ϵ (due to uniform continuity), so by triangle inequality, we can write

$$d(f_n(s), f_n(t)) < N\epsilon \tag{0.15}$$

for some $N \in \mathbb{N}$ which is valid for all $s, t \in (0, 1)$. Now that we know f_n is bounded. We still have

$$f_n(p) - N\epsilon < f_n(x) < f_n(p) + N\epsilon \tag{0.16}$$

since $|f_n(x) - f_n(p)| < N\epsilon$. So the rest of the proof still applies.

Another quick way to see this is to extend f to $\bar{f} : [a, b] \to \mathbb{R}$ as in problem set 2. This extension is unique and uniformly continuous, and with a bit of extra work, we can show that it is uniformly continuous with the same δ as before. Then $\bar{f}_n$ is equicontinuous, and we can apply the exact same proof as before. And if $(\bar{f}_n)$ is uniformly bounded, then so must (f_n) be uniformly bounded.

ii. For $\mathbb{R}$, consider the equicontinuous sequence $f_n(x) = x$ (the functions are all the same). Each f_n is uniformly continuous with $\delta = \epsilon$ (from first-year calculus) and because all the functions are the same, it is equicontinuous.

Also, because the functions are the same, $f_n(p)$ is bounded (as it is just the same point over and over again). However, (f_n) is not uniformly bounded because each individual function is not bounded.

iii. For $\mathbb{Q}, \mathbb{N}$ we can consider the same sequence as before, $f_n(x) = x$. Note that this is still uniformly continuous since the metric is induced from $\mathbb{R}$. Therefore, if $f : M \to \mathbb{R}$ is uniformly continuous, then $f|_S : S \to \mathbb{R}$ is uniformly continuous, where $S \subseteq M$.

So by the same reasons, the preconditions hold, but (f_n) is not uniformly bounded since $f_n(x)$ is not bounded.

3. (a) **Convergence on a countable dense subset:** The proof for this is very similar to the proof of the Arzela-Ascoli Theorem. Since M is compact, consider any countable dense subset $D \in \{d_1, d_2, \dots\} \subseteq M$. Now consider the sequence

$$i_1(d_1), \quad i_2(d_1), \quad i_3(d_1), \quad \dots \tag{0.17}$$

This is a sequence of points in M, so there exists a converging subsequence

$$i_{n_{1,1}}(d_1), \quad i_{n_{2,1}}(d_1), \quad i_{n_{3,1}}(d_1), \quad \dots. \tag{0.18}$$

Now consider the sequence

$$i_{n_{1,1}}(d_2), \quad i_{n_{2,1}}(d_2), \quad i_{n_{3,1}}(d_2), \quad \dots. \tag{0.19}$$

Again, this has a converging subsequence

$$i_{n_{1,2}}(d_2), \quad i_{n_{2,2}}(d_2), \quad i_{n_{3,2}}(d_2), \quad \dots. \tag{0.20}$$

We can repeat this process, in order to create a countable number of converging subsequences, where the isometries involved in each sequence are a subset of the isometries used in the previous sequence. That is, we have:

$$i_{n_{1,1}}(d_1), \quad i_{n_{2,1}}(d_1), \quad i_{n_{3,1}}(d_1), \quad \dots \tag{0.21}$$
$$i_{n_{1,2}}(d_2), \quad i_{n_{2,2}}(d_2), \quad i_{n_{3,2}}(d_2), \quad \dots \tag{0.22}$$
$$i_{n_{1,3}}(d_3), \quad i_{n_{2,3}}(d_3), \quad i_{n_{3,3}}(d_3), \quad \dots \tag{0.23}$$
$$\vdots \tag{0.24}$$

Now consider the sequence of isometries $i_{n_{1,1}}, i_{n_{2,2}}, i_{n_{3,3}}, \dots$. We will show that this sequence converges over D. For any point $d_k \in D$, there exists $N \in \mathbb{N}$ (i.e. $N = k$) such that $j > N$ implies that $i_{n_{j,j}} \subseteq \{i_{n_{1,k}}, i_{n_{2,k}}, \dots\}$, which was defined to send d_k to a convergent sequence.

Convergence on M: We will use the following lemma:

> **Lemma 2:** For any $x \in M$ and $\delta > 0$, we can pick $d_j \in D$ such that $d(d_j, x) < \delta$ and $j \leq J$, where $J \in \mathbb{N}$ is picked large enough such that every $x \in M$ is within δ of some d_j with $j \leq J$.
>
> *Proof.* Because M is compact, it is totally bounded, so we can cover it with a finite number of balls with radius $\delta/2$ for all $\delta > 0$. Every ball contains a point $d_{j_k} \in D$ such that the distance between any point in this ball and d is less than δ.
>
> Pick $J = \max\{d_{j_1}, \dots, d_{j_k}\}$. ☐

For all $\epsilon > 0$ there exists N such that $n_k, n_\ell > N$ implies for all x, we can find a d_j such per the lemma above such that:

$$d(i_{n_k}(x), i_{n_\ell}(x)) \leq d(i_{n_k}(x), i_{n_k}(d_j)) + d(i_{n_k}(d_j), i_{n_\ell}(d_j)) + d(i_{n_\ell}(d_j), i_{n_\ell}(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \tag{0.25}$$

We know that $d(i_{n_k}(x), i_{n_k}(d_j)) = d(i_{n_\ell}(d_j), i_{n_\ell}(x)) < \epsilon/3$ since $d(x, d_j) < \epsilon$ per the lemma above.

Furthermore, we also know that $d(i_{n_k}(d_j), i_{n_\ell}(d_j)) < \epsilon/3$ for large enough N since $i_{n_{k'}}(d_j)$ converges in M, so it must be Cauchy. Note that here, N depends on the choice of d_j, but since from the lemma there are a finite number of d_j's for a given ϵ, we can pick N to pick

Note that there exists an $N \in \mathbb{N}$ such that for all x, we have $d(i_{n_k}(x), i_{n_\ell}(x)) < \epsilon$, so this gives us uniform convergence as well.

Converges to isometry: What remains to be shown is that i_{n_k} converges to an isometry. Consider arbitrary $p, q \in M$. We have shown that $i_{n_k}(p) \to i(p) = p_0$ and $i_{n_k}(q) \to i(q) = q_0$. We must show that $d(p_0, q_0) = d(p, q)$.

We prove this by contradiction. Suppose for the sake of contradiction that

$$|d(i(p), i(q)) - d(p, q)| = \epsilon > 0. \tag{0.26}$$

However, by triangle inequality:

$$d(i(p), i(q)) \leq d(i(p), i_{n_k}(p)) + d(i_{n_k}(p), i_{n_k}(q)) + d(i_{n_k}(q), i(q)) = d(i(p), i_{n_k}(p)) + d(i_{n_k}(q), i(q)) + d(p, q), \tag{0.27}$$

so

$$d(i(p), i(q)) - d(p, q) \leq d(i(p), i_{n_k}(p)) + d(i_{n_k}(q), i(q)). \tag{0.28}$$

However, since $i_{n_k}(p) \to i(p)$ and $i_{n_k}(q) \to i(q)$, there exists some N such that $k > N$ implies that $d(i(p), i_{n_k}(p)) < \epsilon/2$ and $d(i_{n_k}(q), i(q)) < \epsilon/2$. So we have:

$$|d(i(p), i(q)) - d(p, q)| < \epsilon, \tag{0.29}$$

contradicting the assumption that $|d(i(p), i(q)) - d(p, q)| = \epsilon$. Therefore, the sequence of isometries converges to an isometry.

(b) The space of self isometries is compact if any sequence of isometries has a convergent subsequence. We proved this for arbitrary sequences of isometries in part (a), so this space must be compact.

(c) We prove this directly. Consider an arbitrary $x \in M$. We wish to show that

$$i_{n_1}^{-1}(x), \ldots, i_{n_2}^{-1}(x), \ldots \tag{0.30}$$

converges to i^{-1}. To do so, for any $\epsilon > 0$, we want to show there exists $K \in \mathbb{N}$ such that $k > K$ implies that $d(i_{n_k}^{-1}(x), i^{-1}(x)) < \epsilon$. But by the definition of isometries, we have

$$d(i_{n_k}^{-1}(x), i^{-1}x) = d(x, i_{n_k}(i^{-1}(x))) = d(i(i^{-1}x), i_{n_k}(i^{-1}x)) = d(i(p), i_{n_k}(p)), \tag{0.31}$$

where $p = i^{-1}x$. But because i_{n_k} uniformly converges to i, there exists $K \in \mathbb{N}$ such that for all points $p \in M$ we have that $d(i(p), i_{n_k}(p)) < \epsilon$. Therefore, if we pick this same K value for all choices of x, we have

$$d(i_{n_k}^{-1}(x), i^{-1}x) < \epsilon, \tag{0.32}$$

as desired.

(d) See part (e)

(e) Yes, they are compact. The group of $m \times m$ orthogonal matrices is isomorphic to $O(m)$, which defines isometries of $\mathbb{R}^m$ that fixes the origin. It is a standard linear algebra exercise to show that this corresponds to isometries on the unit $m - 1$ sphere. This is a compact space, and we've shown that the space of self-isometries on compact spaces is compact, so $O(m)$ is compact.

4. (a) Consider the function

$$f(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ x^y y^x & \text{else.} \end{cases} \tag{0.33}$$

This is symmetric in x and y, so we just need to check that for each fixed $y = y_0$ the function $g : x \mapsto f(x,y)$ is a continuous function in x. We have two cases:

- Case 1: $y_0 \neq 0$. In this case, we have

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{y_0} y_0^x & \text{else.} \end{cases} \tag{0.34}$$

This is continuous at $x > 0$ because x^{y_0} and y_0^x are both continuous, so their product is continuous. It is also continuous at $x = 0$ because $\lim\limits_{x \to 0^+} x^{y_0} y_0^x = 0$.

- Case 2: $y_0 = 0$. In this case, we have:

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^0 0^x & \text{else.} \end{cases} \tag{0.35}$$

Note that $x^0 0^x = 0$ for all $x \neq 0$, so $g(x) = 0$ everywhere, and it is continuous.

Now, we just need to show that f is not continuous, specifically at $(0,0)$. Consider restricting the function to the set $S = \{(t,t) \in \mathbb{R}^2 : t \in [0,1]\}$. Let this restriction be h. We then have

$$h(t) = \begin{cases} 0 & t = 0 \\ (t^t)^2 & t \neq 0. \end{cases} \tag{0.36}$$

However, this is not continuous at $t = 0$ since $\lim\limits_{t \to 0^+} (t^t)^2 = 1 \neq 0$. Since a restriction of the function on the domain $[0,1] \times [0,1]$ is not continuous, then the function is not continuous.

(b) Consider a sequence (x_n, y_n) that converges in $[0,1] \times [0,1]$. We wish to show that $(f(x_n, y_n))$ also converges, which would imply that f is continuous.

Let (x_n, y_n) converge to (x_0, y_0). Then for all $\delta > 0$, there exists $N \in \mathbb{N}$ such that $n > N \implies |x_0 - x_n| < \delta, |y_0 - y_n| < \delta$.

Let $g(y) = f(x_0, y)$. Because $g(y)$ is continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|y_0 - y_n| < \delta \implies |g(y_0) - g(y_n)| < \dfrac{\epsilon}{2} \implies |f(x_0, y_0) - g(x_0, y_n)| < \dfrac{\epsilon}{2}$.

Let $h_y(x) = f(x,y)$ be the restriction of f to a certain y value. Because these are equicontinuous for each y, we have that $|x_0 - x_n| < \delta \implies |h_{y_n}(x_0) - h_{y_n}(x_n)| < \epsilon \implies |f(x_0, y_n) - f(x_n, y_n)| < \epsilon$.

We can now put everything together. Consider an arbitrary $\delta > 0$, and pick the corresponding $N \in \mathbb{N}$ such that both $|x_0 - x_n| < \delta$ and $|y_0 - y_n| < \delta$ is satisfied for all $n > N$. Then by the triangle inequality, and using the above results, we have:

$$|f(x_0, y_0) - f(x_n, y_n)| \leq |f(x_0, y_0) - f(x_0, y_n)| + |f(x_0, y_n) - f(x_n, y_n)| \tag{0.37}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{0.38}$$

where the last line follows from continuity of $g(y)$ and equicontinuity of $h_y(x)$.

5. Let
$$P_n(x) = a_{10,n}x^{10} + a_{9,n}x^9 + \cdots + a_{0,n}x^0. \tag{0.39}$$
Note that each coefficient $(a_{i,n})$ forms a bounded sequence since $P_n(x)$ converges to 0 on $[0,1]$, so they must be bounded. Because the coefficients are all bounded, we must have that $P_n(x)$ is uniformly bounded on $[0,1]$ and so are the derivatives. Because they are uniformly bounded on a compact set, they are equicontinuous, so we get uniform convergence of $P_n(x)$ and its derivatives.

We just need to prove that $(a_{i,n})$ forms a bounded sequence.

> **Lemma 3:** $(a_{i,n})$ forms a bounded sequence.
>
> *Proof.* These 11 coefficients are uniquely determined by 11 points of $P_n(x)$. Pick 11 arbitrary points in $[0,1]$. Because it converges point-wise at these 11 points, then these coefficients cannot grow without bound.
>
> Note: I couldn't quite finish the proof here, but it makes sense that if any of the coefficients were to grow unbounded, then $P_n(x)$ cannot converge. An alternative way to prove this is to map this to a problem in $\mathbb{R}^{11}$ and match the coefficients to coordinates and in $\mathbb{R}^n$ a point converges to 0 if all the components converge to 0 ☐

By extension, all higher derivatives will be uniformly continuous on $[0,1]$. We can apply this to the problem, where $d = 10$. Suppose
$$P_n(x) = a_{10,n}x^{10} + a_{9,n}x^9 + \cdots + a_{0,n}x^0. \tag{0.40}$$
We can show that all the coefficients approach zero, i.e. $a_{k,n} \to 0$. To do this, the kth derivative is

$$G_{k,n}(x) = \frac{d^k}{dx^k}P_n(x) = c_{10,k}a_{10,n}x^{10-k} + c_{9,k}a_{9,n}x^{9-k} + \cdots + c_{k,k}a_{k,n}, \tag{0.41}$$

where $c_{i,k} = i(i-1)\cdots(i-k+1)$ come from repeated applications of the power rule. But from the lemma, we know that $G_{k,n}(x)$ uniformly approaches 0 on $[0,1]$, so $G_{k,n}(0) \to 0$. But $G_{k,n}(0) = c_{k,k}a_{k,n}$, and since $c_{k,k} > 0$ is a constant that doesn't depend on n, we have that $a_{k,n} \to 0$.

We have shown that all the coefficients approach 0. We will now use this to show that $P_n(x)$ uniformly converges to 0 on the interval $[4,5]$ as well. Because $a_{i,n} \to 0$, we have that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|a_{i,n}| < \dfrac{\epsilon}{11 \cdot 5^{10}}$.

Let $4 \le x \le 5$. By triangle inequality, we have:

$$\begin{aligned}
|P_n(x)| &= |a_{10,n}x^{10} + a_{9,n}x^9 + \cdots + a_{0,n}x^0| && (0.42)\\
&\le |a_{10,n}||x^{10}| + |a_{9,n}||x^9| + \cdots + |a_{0,n}||x^0| && (0.43)\\
&\le |a_{10,n}|5^{10} + |a_{9,n}|5^{10} + \cdots + |a_{0,n}|5^{10} && (0.44)\\
&< \frac{\epsilon}{11} + \cdots + \frac{\epsilon}{11} = \epsilon. && (0.45)
\end{aligned}$$

For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|P_n(x) - 0| < \epsilon$, so the sup-norm approaches 0 and $P_n(x) \to 0$ uniformly on $[4,5]$.