MAT357: Real Analysis
Problem Set 6

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1. (a) Consider $M = [0,1]$ and $f : M \to M$ given by $f(x) = \frac{1}{1+x}$. We can show that this is a weak contraction since for all $x, y \in M$ with $x \neq y$, we have

$$\left|\frac{1}{1+y} - \frac{1}{1+x}\right| = \left|\frac{x-y}{(1+x)(1+y)}\right| \tag{0.1}$$
$$< |x-y|. \tag{0.2}$$

The inequality is strict because the only time when $(1+x)(1+y) = 1$ is when $x = y = 0$, in which case we don't have $x \neq y$. However, this is not a contraction. For any k such that $0 < k < 1$, we can find $x, y \in M$ such that $d(fx, fy) = kd(x, y)$. WLOG, let $y < x$. Then:

$$d(fx, fy) - kd(x, y) = \frac{1}{1+y} - \frac{1}{1+x} - k(x-y) \tag{0.3}$$
$$= (x-y)\left(\frac{1}{(1+x)(1+y)} - k\right). \tag{0.4}$$

We can choose $y = 0$ and $x = \frac{1}{k} - 1$ for this to be zero. Note that for any $0 < k < 1$, we have $x, y \in M$ and $x \neq y$. Therefore,

$$d(fx, fy) = kd(x, y) \tag{0.5}$$

for some $x, y \in M$ with $x \neq y$, so it is not a contraction.

(b) The above example worked it out for a compact space $[0, 1]$.

(c) Because f is a weak contraction, it is Lipschitz continuous, so it is continuous. A point x is a fixed point of f if $d(x, f(x)) = 0$. Recall that $d : M^2 \to \mathbb{R}$ is a continuous function, so

$$g : M \to \mathbb{R}, \qquad g(x) = d(x, f(x)) \tag{0.6}$$

is a composition of continuous function, so g is continuous. Because it is a continuous function defined on a compact metric space, it is bounded and the extreme value theorem applies: there exists some point y such that $g(y) = \inf_{x \in M}\{g(x)\}$.

We claim that $g(y) = 0$, i.e. y is a fixed point. Suppose it is not. Then consider

$$g(f(y)) = d(f(y), f(f(y))) < d(y, f(y)) = g(y), \tag{0.7}$$

where the inequality is due to the fact that f is a weak contraction. This contradicts that $g(y)$ is the minimum of $g(x)$ over all $x \in M$. Therefore, y is a fixed point.

To show the uniqueness of a fixed point is easy. Suppose there were two fixed points $y_1, y_2 \in M$. Then by weak contraction, we want

$$d(f(y_1), f(y_2)) < d(y_1, y_2). \tag{0.8}$$

But this is impossible since $d(f(y_1), f(y_2)) = d(y_1, y_2)$. Therefore, there is only one fixed point.

2. (a) Let us define

$$f(x) = \begin{cases} x + \frac{1}{x} & x \leq -1 \\ -2 & x > -1. \end{cases} \tag{0.9}$$

It is easy to check that function f is continuous at $x = -1$. The derivative is

$$f'(x) = \begin{cases} 1 - \frac{1}{x^2} & x \leq -1 \\ 0 & x > -1. \end{cases} \tag{0.10}$$

It's also easy to verify that $|f'(x)| < 1$. Finally, also notice that $f(x) = x$ has no solutions. For $x < -1$, we can't have $x + \frac{1}{x} = x$. For $x > -1$, we have $f(x) > -2$. Therefore, there are no fixed points, so f is not a contraction.

(b) Yes, it is a weak contraction. Suppose for the sake of contradiction that it is not weak. That is, there exists $a, b \in \mathbb{R}$ where $b > a$ such that

$$\frac{f(b) - f(a)}{b - a} = M. \tag{0.11}$$

where $|M| \geq 1$. By the mean value theorem, there exists some $c \in \mathbb{R}$ with $a < c < b$ such that $f'(c) = M$. But since $|M| \geq 1$ we have $|f'(c)| \geq 1$, contradicting the assumption that $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(c) See part (a). We found a function f that satisfies the preconditions but is not a contraction.

3. (a) Limiting case of part (b), where $C = 0$ and $M = K$.

(b) If $|f(x)| \leq C|x| + K$ for all x, then

$$|x'(t)| \leq C|x| + K \tag{0.12}$$

for all x. We first show that $x(t)$ is bounded above for finite time, and by the same argument (or by symmetry), we can show that $x(t)$ is bounded below.

Suppose for the sake of contradiction that the solution diverges to positive infinity in finite time. Therefore, WLOG, we can let $x > 0$ in our time of interest. This is equivalent to

$$x'(t) \leq Cx + K \iff x'(t) - Cx \leq K \tag{0.13}$$

$$\iff e^{-Ct}x'(t) - Ce^{-Ct}x \leq Ke^{-Ct} \tag{0.14}$$

$$\iff \frac{d}{dt}(e^{-Ct}x) \leq Ke^{-Ct}. \tag{0.15}$$

Integrating over some bounded interval $0 \leq t \leq T$, we have

$$e^{-CT}x(T) - x(0) \leq \int_0^T Ke^{-Ct}\,\mathrm{dt}\,. \tag{0.16}$$

If $C = 0$, then

$$x(T) - x(0) \leq KT \implies x(T) \leq x(0) + KT. \tag{0.17}$$

If $C > 0$, then

$$e^{-CT}x(T) - x(0) \leq \frac{K}{C}\left(e^{-CT} - 1\right) \implies x(T) \leq \left(x(0) + \frac{K}{C}\right)e^{CT} - \frac{K}{C}. \tag{0.18}$$

which is bounded above. Since $T \in \mathbb{R}^+$ is arbitrary, we have shown that $x(t)$ does not diverge in finite t.

Note: There is actually one case we have not considered here. Perhaps it is possible for a solution to diverge in finite time, yet not diverge to either positive or negative infinity. Instead, it oscillates, i.e.

$$x(t) = \frac{\sin(1/(t - T))}{t - T}. \tag{0.19}$$

We will show that this behavior which we neglected, cannot actually exist! This is because this solution would satisfy $x(t) = 0$ at infinitely many times t_i in a bounded interval. However, when $x = 0$ we have

$$|x'(t_i)| \leq K \tag{0.20}$$

so for any $\epsilon > 0$, there exists $\delta > 0$ such that for all i we have $|t - t_i| < \delta$ implies $|x'(t) - K| < \epsilon$. If we choose $\epsilon = K$, we have $|t - t_i| < \delta$ implies

$$-2K < x'(t) < 2K \implies |x(t)| < 2K\delta. \tag{0.21}$$

But the intersection points t_i get arbitrarily close together (i.e. closer than δ), so by the above $x(t)$ must be bounded by a constant $2K\delta$ when this occurs, contradicting the fact that $x(t)$ diverges.

(c) Not assigned.

(d) We just need to show that $f(x)$ being uniformly continuous implies that there are constants C, K such that $|f(x)| \leq C|x| + K$. Intuitively, this is true since uniform continuity states that in a δ-interval, the function can only change by so much. We prove this rigorously below.

Fix any $\epsilon > 0$. Then there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Choose $C = \epsilon/\delta$.

For each $x \in \mathbb{R}$, choose $N = \lceil sx/\delta \rceil$ where $s \in \{+1, -1\}$ is the sign of x. Then by triangle inequality:

$$|f(x) - f(0)| \leq |f(0) - f(s\delta)| + |f(s\delta) - f(2s\delta)| + \cdots + |f(s(N-1)\delta) - f(x)| < N\epsilon, \tag{0.22}$$

where the last inequality comes from applying the $\delta - \epsilon$ definition to each term. Note that

$$N\epsilon < (|x|/\delta + 1)\epsilon = C|x| + \epsilon. \tag{0.23}$$

We have $|f(x) - f(0)| < C|x| + \epsilon$, which we can rewrite as

$$f(0) - C|x| - \epsilon < f(x) < f(0) + C|x| + \epsilon. \tag{0.24}$$

There exists $K > 0$ such that $K > |f(0)| + \epsilon$. Then the above inequality implies that

$$-K - C|x| < f(x) < K + C|x| \implies |f(x)| < K + C|x|, \tag{0.25}$$

which is a stronger version of the desired inequality.

4. (a) Let μ denote the measure defined using open intervals and μ' denote the measure defined using closed intervals. We wish to show that given any set $X \in \mathbb{R}$ we have $\mu(X) = \mu'(X)$. First note that $|(a, b)| = |[a, b]| = b - a$, so the length of an open interval and a closed interval with the same boundary is the same.

Note that $\mu(X) \geq \mu'(X)$ is easy to show since if we have an open interval covering of X, we can take the closure of each interval to get a closed interval covering of X. The length of the covering remains unchanged. Therefore, the infimum of the length of a closed interval covering is at most the infimum of the length of an open interval covering.

To show that $\mu'(X) \geq \mu(X)$, we consider a closed interval covering of X with length L. We can then construct an open interval covering of X with length $L + \epsilon$ for any $\epsilon > 0$. To do so, consider an arbitrary closed covering and construct an open covering,

$$\bigcup_{i=1}[a_i, b_i] \subset \bigcup_{i=1}(a_i - \epsilon/2^{i+1}, a_i + \epsilon/2^{i+1}), \tag{0.26}$$

where the total length is

$$\sum_{i=1}(a_i - \epsilon/2^{i+1}, a_i + \epsilon/2^{i+1}) = \sum_{i=1}[a_i, b_i] + \epsilon \sum_{i=1} \frac{1}{2^i} = L + \epsilon. \tag{0.27}$$

Thus, every closed covering can be covered by an open covering with marginally larger length. Therefore, the infimum of the length of an open interval covering is at most the infimum of the length of a closed interval covering.

(b) We can write

$$C = \bigcap_{i}^{\infty} C_k \tag{0.28}$$

where C_k is a closed set with measure $(2/3)^k$. We have $C \subseteq C^k$ for all k so

$$\mu(C) \leq \mu(C^k) \tag{0.29}$$

for all k. But $\mu(C^k) = (2/3)^k$ can be arbitrarily small, so $\mu(C^k) = 0 \implies \mu(C) = 0$.

(c) Let μ denote the measure defined using open intervals, μ' denote the measure defined using closed intervals, and μ'' denote the measure defined using closed and open intervals.

Showing $\mu''(X) \geq \mu'(X)$ is easy, and in fact it follows the exact same steps as showing $\mu(X) \geq \mu'(X)$. We can take any covering (using closed and/or intervals), take the closure of these intervals, and the length of the covering remains unchanged.

Showing $\mu''(X) \geq \mu(X)$ is also easy, and follows the exact same steps as showing $\mu'(X) \geq \mu(X)$. Consider any covering with length L consisted of closed and/or open intervals, and we can construct an open covering where the total length is $L + \epsilon$ for any $\epsilon > 0$ using the same method as in part (a).

However, $\mu'(X) = \mu(X)$, so we really have $\mu(X) \leq \mu''(X) \leq \mu(X)$. Therefore, $\mu(X) = \mu''(X)$.

(d) Let μ denote the measure defined using rectangles and μ' denote the measure using squares. We wish to show that $\mu(X) = \mu'(X)$ for any $X \subseteq\in \mathbb{R}^2$.

First note that $\mu(X) \leq \mu'(X)$ is easy to show since every square-covering is a rectangle-covering.

We wish to show that $\mu'(X) \leq \mu(X)$. For each rectangle, we wish to cover it with a finite number of squares[1] such that the difference in areas can be bounded by ϵ for any $\epsilon > 0$. Suppose this was possible (we will prove it later). Then:

$$\bigcup_{i=1}[a_i, b_i] \times [c_i, d_i] \subset \bigcup_{i=1}\left(\bigcup_{j=1}^{n_i}[x_j, x_j + L_j] \times [y_j, y_j + L_j]\right) \tag{0.30}$$

where the total area is

$$\sum_{i=1}\sum_{j=1}^{n_j} L_j^2 = \sum_{i=1}(b_i - a_i)(c_i - d_i) + \sum_{i=1}\epsilon/2^{i+1} \tag{0.31}$$

$$= \sum_{i=1}(b_i - a_i)(c_i - d_i) + \epsilon. \tag{0.32}$$

[1] Technically it could be a countable number of squares, but then the proof would require accepting the axiom of choice.

Now, it just remains to show that we can cover any rectangle with a finite number of squares such that the difference in areas can be bounded by ϵ. To do so, WLOG, let $R = [0, a] \times [0, b]$ and cover it with squares of side length $\frac{\epsilon}{2^n}$, i.e.

$$R \subset \bigcup_{i=1}^{\text{ceil}(a2^n/\epsilon)} \bigcup_{j=1}^{\text{ceil}(b2^n/\epsilon)} [x_i, x_i + \frac{\epsilon}{2^n}] \times [y_j, y_j + \frac{\epsilon}{2^n}] \tag{0.33}$$

where $x_1 = y_1 = 0$. The difference in area is bounded by $\frac{\epsilon}{2^n}(a + b)$. As n increases, the difference in area can be made arbitrarily small, and we're done.

5. (a) See part (b)

   (b) We first solve an easier problem. Let $S \cong \mathbb{R}^k$ be a hyperplane in $\mathbb{R}^n$ with $k < n$ such that:

   - S passes through the origin
   - Any $v \in S$ is parallel to one of the coordinate axes.

   Then S is measure zero in $\mathbb{R}^n$. A direct consequence is that any $v \in S$ will be orthogonal to one of the coordinate axes, which WLOG we will assume is x_1. Then consider the covering

$$\bigcup_{i=1}^{\infty} U_i \tag{0.34}$$

   where

$$U_i = [-\epsilon/2^{i+1}, +\epsilon/2^{i+1}] \times [-i/2, i/2] \times \cdots \times [-i/2, i/2] \tag{0.35}$$

   which has volume

$$\text{vol}(U_i) = \frac{\epsilon}{2^i} \cdot i^{n-1}. \tag{0.36}$$

   First note that this is a valid covering. Any point $(0, x_2, \ldots, x_n)$ will be covered by $U_{2 \cdot \text{ceil}(\max\{|x_2|, \ldots, |x_n|\})}$. Now, we wish to show that the total volume is bounded by ϵ. We have,

$$\sum_{i=1}^{\infty} \text{vol}(U_i) = \frac{\epsilon i^{n-1}}{2^i} = M\epsilon \tag{0.37}$$

   for some constant $M > 0$. This infinite series converges by the comparison test (standard first-year calculus exercise). Therefore, we can make the volume of the covering arbitrary small by changing ϵ.

   Now we will show that any hyperplane $S' \cong \mathbb{R}^k$ that passes through the origin (with axes not necessarily parallel to coordinate axes) is measure 0. To do this, note that we can write

$$S' = L(S) \tag{0.38}$$

   where S is the hyperplane we originally described, and $L \in SO(n)$ is a linear transformation. Note that $\det(L) = 1$. Therefore, if $\bigcup_{i=1}^{\infty} U_i$ is a cover for S then

$$L\left(\bigcup_{i=1}^{\infty} U_i\right) \supset S' \tag{0.39}$$

   is a cover for S' with the same volume (since $\det(L) = 1$). Therefore, if the measure defined by rotated prisms is equivalent to the measure defined by non-rotated prisms, then S' will have measure 0.

   The proof for this is a bit ugly, but very similar to question 4, and I believe it is something we covered in MAT257 when defining integration on $\mathbb{R}^n$ through defining partitions and taking refinements of these partitions until the error becomes arbitrarily small. This allows us to cover any connected open set with finitely many prisms such that the difference in volumes is bounded by ϵ.

   Finally, we need to show that an arbitrary hyperplane S'' is measure 0. To do this, note that we can perform a constant shift such that this hyperplane intersects the origin and becomes S'. We proved that S' is measure 0, so we can take a cover for S' and shift it by the same constant to get a cover for S'' such that the cover has the same volume. Therefore, S'' is measure 0.