PHY365: Quantum Information
Midterm #1 Review

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Contents

1 Quantum Gates

All quantum gates are unitary matrices, i.e. $U^\dagger U = UU^\dagger = I$, and in general can be represented as

$$\hat{U} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}. \tag{1.1}$$

Some common gates:

- The Pauli gates $\hat{X}, \hat{Y}, \hat{Z}$ are given by

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- And the Hadamard gate

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1.2}$$

  sends $|0\rangle \mapsto |+\rangle$ and $|1\rangle \mapsto |-\rangle$.

2 Measurements

Quantum measurements are described by a collection $\{M_m\}$ of **measurement operators.** These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement, then the probability that result m occurs is given by

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle . \tag{2.1}$$

and the state of the system after the measurement is given by

$$\frac{M_m |\psi\rangle}{\sqrt{p(m)}}. \tag{2.2}$$

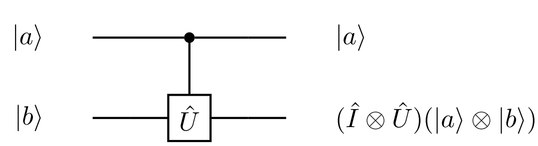
The measurement operators satisfy the **completeness equation** (fancy talk for probabilities summing to one):

$$\sum_m M_m^\dagger M_m = I. \tag{2.3}$$

To measure in a basis $|m\rangle$, where $|m\rangle$ forms an orthonormal basis, simply means to perform the projective measurement with projectors $P_m = |m\rangle \langle m|$. Note in lecture, we used Π as the symbol (but I believe P is more commonly used).

3 Quantum Circuits

A control gate is in the form of:



where the dot represents the **control.** In a sense, we apply $\hat{U}$ if and only if the control is 1.

4 Schmidt Decomposition Theorem

Any two-qubit pure state can be written as

$$|\Psi\rangle = \hat{U}_A \otimes \hat{U}_B \left(\lambda_0 |00\rangle + \lambda_1 11\right), \tag{4.1}$$

where λ_0, λ_1 are real, positive constants known as **singular values** and they satisfy $\lambda_0^2 + \lambda_1^2 = 1$. The operators $\hat{U}_A, \hat{U}_B$ are unitaries applied separately to each qubit.

The unitary operators $\hat{U}_A, \hat{U}_B$ are given by unitary matrices that satisfy the relationships

$$\hat{\chi}\hat{\chi}^\dagger = \hat{U}_A \Lambda^2 \hat{U}_A^\dagger, \qquad \hat{\chi}^\dagger \hat{\chi} = \hat{U}_B \Lambda^2 \hat{U}_B^\dagger. \tag{4.2}$$

where χ is a matrix where entries are the coefficients of $|\Psi\rangle$ and $\Lambda^2 = \text{diag}(\lambda_0^2, \lambda_1^2)$, where λ_i are solutions to the quadratic equation:

$$\lambda^4 - \lambda^2 + (C/2)^2 = 0, \tag{4.3}$$

where C is the concurrence.

5 Entangled States

The **concurrence** of $\alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle$ is given by

$$C = 2|\alpha\delta - \beta\gamma| \tag{5.1}$$

where it is maximally entangled if $C = 1$ and separable if $C = 0$.

The **fundamental theorem of entanglement** says that for a $C = 1$ two-qubit system,

$$(\hat{I} \otimes \hat{U}) |\beta\rangle = -(\hat{U}^\dagger \otimes \hat{I} |\beta\rangle) \tag{5.2}$$

and

$$(\hat{U} \otimes \hat{U}) |\beta\rangle = -|\beta\rangle \tag{5.3}$$

The **Bell States** are four maximally entangled two qubit states that form a basis for the four-dimensional Hilbert space for two qubits. They are given by:

$$\begin{aligned}
|\beta_0\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |\Phi\rangle \\
|\beta_1\rangle &= \frac{i}{\sqrt{2}} (|01\rangle + |10\rangle) = i |\Psi_+\rangle \\
|\beta_2\rangle &= \frac{-1}{\sqrt{2}} (|01\rangle - |10\rangle) = -i |\Psi_-\rangle \\
|\beta_3\rangle &= \frac{i}{\sqrt{2}} (|00\rangle - |11\rangle) = -|\Phi_-\rangle
\end{aligned}$$