PHY365: Quantum Information
Problem Set 2

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1. (a) Suppose $|\varphi\rangle = a\,|0\rangle + b\,|1\rangle$. Then we can compute

$$S_i = \langle\varphi|\hat{\sigma}_i|\varphi\rangle \tag{0.1}$$

for $i = 1, 2, 3.$

- $S_1 = ab^* + a^*b = 2\,\mathrm{Re}(ab^*)$
- $S_2 = i(ab^* - ab^*) = 2\,\mathrm{Im}(ab^*)$
- $S_3 = |a|^2 - |b|^2$

Also note that $S_1^2 + S_2^2 + S_3^3 = 1.$

(b) Apply the rules in part (a).

(c) i. $|\psi\rangle\langle\psi| = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$. The eigenvalue equation is given by

$$\left(|a|^2 - \lambda\right)\left(|b|^2 - \lambda\right) - ab^*ba^* = 0 \implies \left(|a|^2 - \lambda\right)\left(|b|^2 - \lambda\right) = |a|^2|b|^2 \tag{0.2}$$

One eigenvalue is $\lambda = 0$. Assuming $\lambda \neq 0$, we can divide through to get

$$-|a|^2 - |b|^2 + \lambda = 0 \implies \lambda = |a|^2 + |b|^2 = 1. \tag{0.3}$$

Therefore, the eigenvalues are $\lambda = 0, 1.$

ii. We can compute:

$$\hat{I} + \boldsymbol{S}\cdot\boldsymbol{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\,\mathrm{Re}(ab^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2\,\mathrm{Im}(ab^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \left(|a|^2 - |b|^2\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us look at each element of $\frac{1}{2}(\hat{I} + \boldsymbol{S}\cdot\boldsymbol{\sigma})$ individually:

$$\begin{aligned} c_{11} &= \frac{1}{2} + \frac{|a|^2 - |b|^2}{2} = \frac{1 + |a|^2 - 1 + |a|^2}{2} = |a|^2 \\ c_{12} &= \mathrm{Re}(ab^*) - i\,\mathrm{Im}(ab^*) = (ab^*)^* = a^*b \\ c_{21} &= \mathrm{Re}(ab^*) + i\,\mathrm{Im}(ab^*) = ab^* \\ c_{22} &= \frac{1 - |a|^2 + |b|^2}{2} = |b|^2 \end{aligned}$$