PHY365: Quantum Information
Problem Set 3

QiLin Xue

Winter 2022

1. We can directly compute:
$$C = 2|\alpha\delta - \beta\gamma| = 0.96 \tag{0.1}$$

2. We compute λ_0^2, λ_1^2 to be
$$\lambda^4 + \lambda^2 + C^2/4 = 0 \implies \lambda^2 = 0.36, 0.64. \tag{0.2}$$

   Therefore,
$$\Lambda^2 = \frac{1}{25}\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}. \tag{0.3}$$

   Consider
$$\chi = \begin{pmatrix} i\sqrt{27} & 3 \\ -4 & -i\sqrt{48} \end{pmatrix}, \tag{0.4}$$

   so
$$\chi^\dagger = \begin{pmatrix} -i\sqrt{27} & -4 \\ 3 & i\sqrt{48} \end{pmatrix}. \tag{0.5}$$

   We can construct the hermitian matrices,
$$\hat{F} = \chi\chi^\dagger = \frac{4}{100}\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{0.6}$$
$$\hat{G} = \chi^\dagger\chi = \frac{1}{100}\begin{pmatrix} 43 & 7i\sqrt{3} \\ (-7i)\sqrt{3} & 57 \end{pmatrix}. \tag{0.7}$$

   We can rewrite $\hat{F}$ as:
$$\chi\chi^\dagger = \hat{I}\Lambda^2\hat{I}^\dagger. \tag{0.8}$$

   Rewriting $\hat{G}$ is harder. Its eigenvectors are $v_1 = (-i\sqrt{3}, 1), v_2 = (i/\sqrt{3}, 1)$ for λ_0, λ_1 respectively. We want to diagonalize $\hat{G}$ into the form $V\Lambda^2V^{-1}$, but ensure V is unitary. To do so, we can multiply the eigenvectors by a phase e^{ia} and e^{ib} and scale them to get the following matrix of eigenvectors:
$$V = \frac{\sqrt{3}}{2}\begin{pmatrix} -ie^{ia} & i/\sqrt{3}e^{ib} \\ e^{ia}/\sqrt{3} & e^{ib}. \end{pmatrix} \tag{0.9}$$

   It is straightforward to verify that the determinant has absolute value of 1. To ensure V is unitary, we require the main diagonal entries to be complex conjugates of each other:
$$\begin{aligned} -ie^{ia} &= \left(e^{ib}\right)^* \\ -ie^{i(a+b)} &= 1 \\ \sin(a+b) &= 1, \end{aligned}$$

   as well as the anti-diagonal entries to be the negative of the other's complex conjugate:
$$\begin{aligned} (e^{ia})^* &= -(ie^{ib}) \\ e^{-ia} &= ie^{ib}. \end{aligned}$$

This condition yields both $\cos(a) = \sin(b)$ and $\cos(b) = \sin(a)$. We have three equations, with one solution being $a = b = \pi/4$. Let us denote $\theta = \pi/4$. Our matrix is then:

$$V = \frac{\sqrt{3}}{2} \begin{pmatrix} -ie^{i\theta} & \frac{1}{\sqrt{3}} ie^{i\theta} \\ \frac{1}{\sqrt{3}} e^{i\theta} & e^{i\theta} \end{pmatrix}, \tag{0.10}$$

and we can verify that $\hat{G} = V\Lambda^2 V^{-1}$ and V is unitary. Therefore, we can write:

$$\chi = \hat{I}\Lambda\hat{V}^\dagger = \frac{\sqrt{3}}{10} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -ie^{i\theta} & \frac{1}{\sqrt{3}} ie^{i\theta} \\ \frac{1}{\sqrt{3}} e^{i\theta} & e^{i\theta} \end{pmatrix}^\dagger, \tag{0.11}$$

which is correct up to some phase shift for each term. We can read off the coefficients for $\hat{I}$ and $\hat{V}$ to get the desired

- $a_0 = 1$
- $b_0 = 0$
- $c_0 = -\frac{\sqrt{3}}{2} ie^{i\pi/4}$
- $d_0 = \frac{1}{2} ie^{i\pi/4}$
- $a_1 = 0$
- $b_1 = 1$
- $c_1 = \frac{1}{2} e^{i\pi/4}$
- $d_1 = \frac{\sqrt{3}}{2} e^{i\pi/4}$