PHY450: Relativistic Electrodynamics
Review

QiLin Xue

Spring 2022

Contents

1 Electromagnetic Waves

1.1 Maxwell's Equations

Maxwell's equations in differential form are given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{1.1.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{Faraday} \tag{1.1.2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{Ampere-Maxwell} \tag{1.1.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law} \tag{1.1.4}$$

1.2 Scalar and Vector Potentials

We can write

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{1.2.1}$$

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi \tag{1.2.2}$$

where ϕ is the scalar potential and $\boldsymbol{A}$ is the vector potential.

1.3 Gauge Invariance

The choice of $\boldsymbol{A}$ and ϕ are not unique. The transformations

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi \quad (1.3.1)$$
$$\phi \to \phi' = \phi - \frac{\partial \chi}{\partial t}. \quad (1.3.2)$$

These lead to the same $\boldsymbol{E}' = \boldsymbol{E}, \boldsymbol{B}' = \boldsymbol{B}$. The **Lorenz Gauge** is given by

$$\partial_\mu A^\mu = 0. \quad (1.3.3)$$

and the **Coulomb Gauge** is given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \quad (1.3.4)$$

1.4 The Wave Equation

If we use the Lorenz gauge, we can write the vector potential as a wave equation

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}. \quad (1.4.1)$$

We can write down wave equations for $\boldsymbol{B}, \boldsymbol{E}$ by computing $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B})$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E})$ in a vacuum.

The electric field and magnetic field plane waves can be written in the form of

$$\boldsymbol{E} = \mathrm{Re}\left\{\mathcal{E}_0 \exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right]\right\} \quad (1.4.2)$$
$$\boldsymbol{B} = \mathrm{Re}\left\{\mathcal{B}_0 \exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right]\right\}, \quad (1.4.3)$$

where $\mathcal{E}_0, \mathcal{B}_0$ are complex amplitudes and $\omega = c|\boldsymbol{k}|$ is the frequency. One fundamental idea is that solutions to Maxwell's equations must obey the wave equation, the converse is not true. In fact, from $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ we have that electromagnetic plane waves are perpendicular to the direction of propagation. We have,

$$\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} = 0, \quad (1.4.4)$$

and the magnetic and electric fields are related via

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{k} \times \boldsymbol{E}. \quad (1.4.5)$$

The electric and magnetic fields in a monochromatic plane wave with propagation vector $\boldsymbol{k}$ and polarization $\hat{bmn}$ are given by

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)\hat{\boldsymbol{n}} \quad (1.4.6)$$
$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c}E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}) \quad (1.4.7)$$

1.5 Energy and Momentum in Electromagnetic Waves

The energy per unit volume in an electromagnetic field is given by

$$u = \frac{1}{2}\epsilon_0 \boldsymbol{E}^2 + \frac{1}{2}\mu_0 \boldsymbol{B}^2. \tag{1.5.1}$$

The Poynting vector gives the energy flux density (energy per unit area, per unit time) as

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}. \tag{1.5.2}$$

For a monochromatic plane wave propagating in the $\hat{\boldsymbol{n}}$ direction, we have

$$\boldsymbol{S} = cu\hat{\boldsymbol{n}}. \tag{1.5.3}$$

The average energy per unit volume is given by

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2. \tag{1.5.4}$$

2 Field Theory

2.1 Basic Action

The action for a particle in an electromagnetic field is given by

$$S = S_{\mathsf{free}} + S_{\mathsf{em}} = -mc^2 \int \frac{1}{\gamma}\,\mathrm{d}t + q\int A_\mu \,\mathrm{d}x^\mu + \frac{1}{c}\int j^\mu A_\mu \mathrm{d}^4 x. \tag{2.1.1}$$

2.2 Deriving Lorentz Force Law

Neglecting the field interaction term, we can write the action as

$$S = \int_a^b -mc^2\sqrt{1-u^2/c^2} + q\boldsymbol{A}\cdot\boldsymbol{u} - q\phi \,\mathrm{d}t\,.$$

Minimizing this action will give us the Lorentz Force Law. Specifically,

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}_i} &= \underbrace{m\gamma u_i}_{p_i} + qA_i \\ \frac{\partial L}{\partial x_i} &= q\frac{\partial A_j}{\partial x_i}u_j - q\frac{\partial \phi}{\partial x_i}.\end{aligned}$$

There are two important properties:

$$\begin{aligned}\frac{\partial a_j}{\partial x_i} &= \frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \\ &= \frac{\partial a_i}{\partial x_j} + \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right)\frac{\partial a_m}{\partial x_\ell} \\ &= \frac{\partial a_i}{\partial x_j} + \epsilon_{ijm}\epsilon_{k\ell m}\frac{\partial a_m}{\partial x_\ell}.\end{aligned}$$

This gives us

$$\begin{aligned}\frac{\partial a_j}{\partial x_i}b_j &= \left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\right)\boldsymbol{a} + b_j\epsilon_{ijk}\epsilon_{klm}\frac{\partial a_m}{\partial x_\ell} \\ &= (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} + \left(\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a})\right)_i.\end{aligned}$$

This gives us (using the E-L equation)

$$\frac{d}{dt}(\boldsymbol{p}+q\boldsymbol{A}) = q(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{A} + q\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) - q\boldsymbol{\nabla}\phi.$$

Using the fact that $\frac{d}{dt}\boldsymbol{A} = \frac{\partial A}{\partial t} + (u\boldsymbol{\nabla})\boldsymbol{A}$ we can solve for $\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}$ to get

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \tag{2.2.1}$$

where

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - q\boldsymbol{\nabla}\phi$$

was used.

2.3 Faraday Tensor Motivation

The basic action can be written in the form

$$S = \int_a^b \left(\mathcal{E} - mc^2\right) \mathrm{d}\tau + qA_\mu \,\mathrm{d}x^\mu$$

since $\mathcal{E} = \gamma mc^2$ and $\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma}$. Consider now a variation of the 4-trajectory $x^\mu(\tau) \mapsto x^\mu(\tau) + \delta x^\mu(\tau)$ where $\delta x^\mu(a) = \delta x^\mu(b) = 0$. We can compute the variation in S and set it to zero. That is,

$$\begin{aligned}
\delta S &= \int_a^b -mc^2 \,\mathrm{d}(\delta\tau) + q(\delta A_\mu)\,\mathrm{d}x^\mu + qA_\mu \,\mathrm{d}(\delta x^\mu) \\
&= \int_a^b -mc^2 \left(-\frac{1}{c^2}\right) \eta_\nu \,\mathrm{d}(\delta x^\nu) + q\partial_\nu A_\mu \delta x^\nu \,\mathrm{d}x^\mu + qA_\mu \,\mathrm{d}(\delta x^\mu) \\
&= \int_a^b (m\eta_\nu + qA_\nu)\,\mathrm{d}(\delta x^\nu) + q\partial_\nu A_\mu \delta x^\nu \,\mathrm{d}x^\mu \,.
\end{aligned}$$

Here, we used the fact that

$$\delta A_\mu = \partial_\nu A_\mu \delta x^\nu$$

and

$$\mathrm{d}(\delta\tau) = -\frac{1}{c^2}\eta_\nu(\mathrm{d}(\delta x^\nu))$$

which is derived by taking the variation of both sides of $-c^2 \,\mathrm{d}\tau^2 = \mathrm{d}x^\nu \,\mathrm{d}x_\nu$. Integration by parts on the first term gives us

$$\delta S = \int_a^b \left\{-\mathrm{d}(m\eta_\nu + qA_\nu) + q\partial_\nu A_\mu \,\mathrm{d}x^\mu\right\} \delta x^\nu.$$

Recall that the canonical momentum is $P_\nu = p_\nu + qA_\nu$. We can perform the change of variables:

$$\begin{aligned}
\mathrm{d}\eta_\nu &= \frac{\mathrm{d}\eta_\nu}{\mathrm{d}\tau}\,\mathrm{d}\tau \\
\mathrm{d}A_\mu &= \partial_\mu A_\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau = (\eta^\nu \partial_\mu A_\nu)\,\mathrm{d}\tau \\
\mathrm{d}x^\mu &= \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau = \eta^\mu \,\mathrm{d}\tau \,.
\end{aligned}$$

This gives

$$\delta S = \int_a^b \left\{-m\frac{d\eta_\nu}{\mathrm{d}\tau} - q\eta^\mu \partial_\mu A_\nu + q\eta^\mu \partial_\nu A_\mu\right\} \delta x^\nu \,\mathrm{d}\tau \,.$$

The principle requires that $\delta S = 0$ for the actual trajectory that x^ν takes. Setting this to zero gives

$$m\frac{d\eta_\mu}{\mathrm{d}\tau} = q\eta^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right),$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{2.3.1}$$

is the Faraday field tensor (electromagnetic tensor). Note that

$$m\frac{d\eta_\mu}{d\tau} = q\eta^\nu F_{\mu\nu} \tag{2.3.2}$$

is known as the relativistic form of the Lorentz force.

2.4 Maxwell's Equations from Faraday Tensor

Because $F_{\mu\nu}$ is antisymmetric, we have the Bianchi Identity,

$$\epsilon^{\alpha\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0.$$

Setting $\alpha = 0$ gives

$$\epsilon^{0ijk}\partial_i F_{jk} = \epsilon^{ijk}\partial_i(\epsilon_{jkp}B^p) = 2\partial_i B^i = 2\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0.$$

Setting $\alpha = i$ gives

$$\epsilon^{ij0k}\partial_j F_{0k} + \epsilon^{ijk0}\partial_j F_{k0} + \epsilon^{i0jk}\partial_0 F_{jk} = 0 \implies \epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0,$$

which is Faraday's Law. To get **Ampere-Maxwell** and **Gauss's Law** we need to construct the action for the electromagnetic field and how it interacts with matter.

$$S_{\text{full}} = \int \mathrm{d}^4x\mathcal{L}(A_\nu, \partial_\mu A_\nu), \qquad \mathcal{L}(A_\nu, \partial_\mu A_\nu) = \frac{1}{c}j^k A_k - \frac{1}{4}\epsilon_0 c F_{\kappa\lambda}F^{\kappa\lambda}. \tag{2.4.1}$$

Minimizing using the Euler-Lagrange equations gives us

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu. \tag{2.4.2}$$

Setting $\nu = 0$ gives Gauss's Law and setting $\nu = i$ gives Ampere's Law.

2.5 Noether's Theorem and Stress Energy Tensor

TBA. See pg 13-15 (of the actual book)

2.6 Field Transformations

The total charge and dipole moment of the charge content distribution is

$$Q = \int_V \rho(\boldsymbol{r}', t)\mathrm{d}^3r = \sum_m q_m$$

$$\boldsymbol{d} = \int_V \boldsymbol{r}'\rho(\boldsymbol{r}', t)\mathrm{d}^3r' = \sum_m q_m\boldsymbol{r}_m$$

and the potential is a sum of the static coulomb potential, static dipole moment potential, and the oscillating dipole term.

$$\phi(r\boldsymbol{n}, t) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\boldsymbol{n}\cdot\boldsymbol{d}(t - r/c)}{4\pi\epsilon_0 r^2} + \frac{\boldsymbol{n}\cdot\dot{\boldsymbol{d}}(t - r/c)}{4\pi\epsilon_0 rc} + \mathcal{O}(r'/r)^2$$

and the magnetic vector potential

$$A(r\boldsymbol{n}, t) = \frac{\mu_0}{4\pi}\dot{\boldsymbol{d}}(t - r/c) + \mathcal{O}(r'/r)^2$$

2.7 Radiation