PHY460: Nonlinear Physics
Problem Set 1

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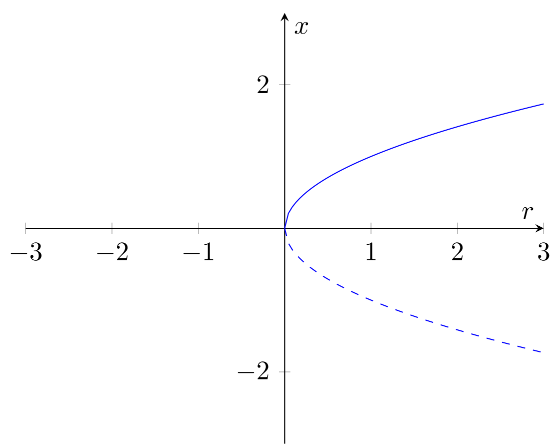
Problem One

3.4.5: We have $\dot{x} = r - 3x^2$. Using the change of variables $x \to X/3$ and $r \to R/3$, we get

$$\frac{1}{3}\dot{X} = \frac{1}{3}R - \frac{1}{3}X^2$$
$$\dot{X} = R - X^2,$$

which is the normal form for a saddle-node bifurcation, so we have bifurcation at $R = 0 \implies r = 0$. The bifurcation diagram is

Bifurcation Diagram of 3.4.5



3.4.6 Let us find the fixed points. Setting $\dot{x} = 0$, we get

$$0 = rx - \frac{x}{1+x}$$
$$\implies r = \frac{1}{1+x}$$
$$\implies x = \frac{1}{r} - 1,$$

where we factored out the second solution of $x = 0$. For every value of $r \neq 1$, we have two distinct fixed points, so a bifurcation could only occur at $r_c = 1$, where they cross, and are both $x^* = 0$. For small variations in x around (r_c, x^*), we have

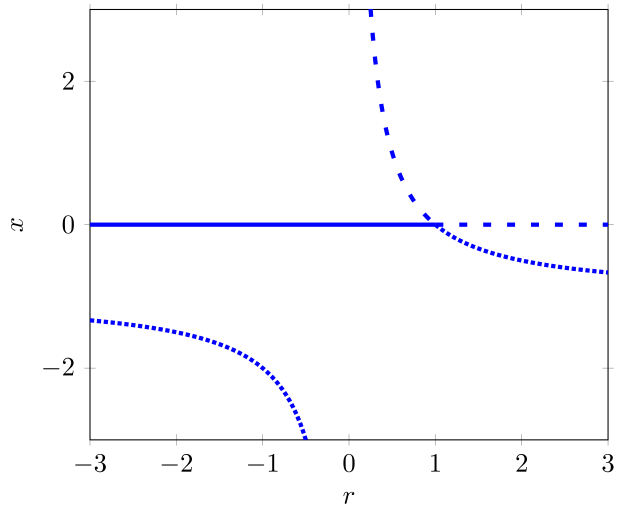
$$\dot{x} = rx - x\left(1 - x + \mathcal{O}(x^2)\right)$$
$$= rx - x + x^2 - \mathcal{O}(x^3)$$
$$= (r-1)x + x^2 - \mathcal{O}(x^3).$$

Making the substitution $r \to R + 1$ and $x \to X$, and ignoring higher order terms, we have

$$\dot{X} = RX + X^2,$$

which is the normal form for a transcritical bifurcation, which occurs at $R = 0 \implies r = 1$. Here, densely dotted lines represent semi-stable fixed points and loosely dashed lines represent unstable fixed points. The bifurcation diagram is

Bifurcation Diagram of 3.4.6



3.4.7: Let us find the fixed points. We have

$$\begin{aligned} 0 &= 5 - re^{-x^2} \\ \implies e^{-x^2} &= \frac{5}{r} \\ \implies -x^2 &= \ln\left(\frac{5}{r}\right), \end{aligned}$$

which could have two solutions if $\frac{5}{r} < 1$, no solutions if $\frac{5}{r} > 1$, and one solution if $r = 1$, which corresponds to $x^* = 0$. Thus, $r_c = 1$ is a bifurcation, and we can classify what type of bifurcation by considering small variations in x around (r_c, x^*). We have

$$\begin{aligned} \dot{x} &= 5 - r\left(1 - x^2 + \mathcal{O}(x^4)\right) \\ &= 5 - r + rx^2 - \mathcal{O}(x^4). \end{aligned}$$

Making the substitution $x \to X/r$ and $r \to r(5-r)$, ignoring higher order terms, we have

$$\begin{aligned} \frac{1}{r}\dot{X} &= 5 - r + \frac{X^2}{r} \\ \implies \dot{X} &= r(5-r) + X^2 \\ \implies \dot{X} &= R + X^2, \end{aligned}$$

which is a saddle-node bifurcation. Note that the change of variables

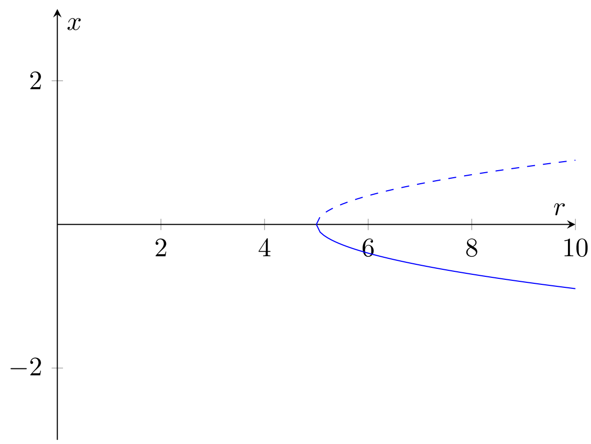
$$(r, x) \leftrightarrow (r(5-r), rx)$$

is not a regular scaling of variables, but at least locally, this is a diffeomorphism since the Jacobian is

$$\begin{aligned} |J| &= \det\begin{pmatrix} 5 - 2r & 0 \\ x & r \end{pmatrix} \\ &= |(5-2r)r| \end{aligned}$$

which at the bifurcation at $r = 5$, gives $|J| = 25$, which shows it is invertible. Therefore, in some small neighborhood of $(r_c, x^*) = (1, 0)$, the change of variables is linear. The bifurcation diagram is

Bifurcation Diagram of 3.4.7



3.4.8: Let us find the fixed points. We have

$$\begin{aligned} 0 &= rx - \frac{x}{1+x^2} \\ \implies r &= \frac{1}{1+x^2} \\ \implies x^2 &= \frac{1}{r} - 1, \end{aligned}$$

where we factored out the solution $x = 0$.

- $r \leq 0$ has 1 fixed point at $x = 0$
- $0 < r < 1$ has three fixed points, one at $x = 0$ and two at $\pm\sqrt{1/r - 1}$.
- $r \geq 1$ has one fixed at $x = 0$.

Therefore, we have three bifurcations, one at $r = 0$ and one at $r = 1$. We start off at $r_c = 1$ which has $x_c = 0$. For small variations in x, we have

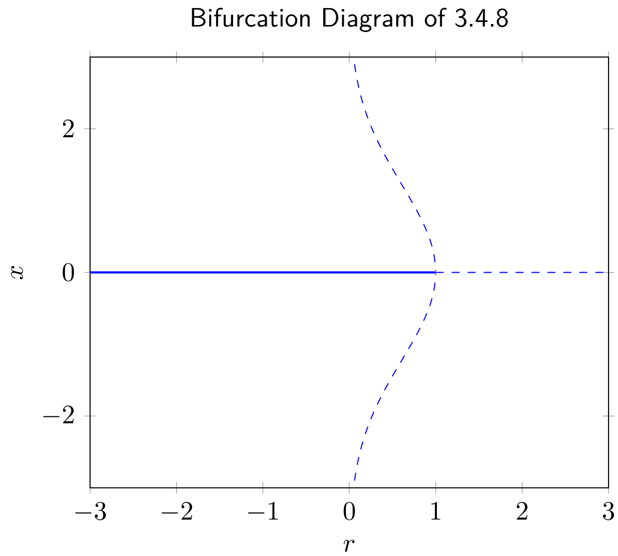
$$\begin{aligned} \dot{x} &= rx - x\left(1 - x^2 + \mathcal{O}(x^4)\right) \\ &= rx - x + x^2 + \mathcal{O}(x^5) \\ &= (r-1)x + x^2 + \mathcal{O}(x^5). \end{aligned}$$

Using the change of variables $r \to R + 1$ and $x \to X$, and ignoring higher order terms, we have

$$\dot{X} = RX + X^2,$$

which is a sub-critical pitchfork bifurcation. The $r = 0$ bifurcation is tougher because as we approach $r = 0$ from the positive side, the critical points diverge to $\pm\infty$, so there is no single-fixed point we can analyze around, yet $r = 0$ is clearly a bifurcation since the behavior clearly changes at $r = 0$ (from one fixed point to three fixed points). To analyze this type of special bifurcation, we have not yet learned the necessary tools to do so.

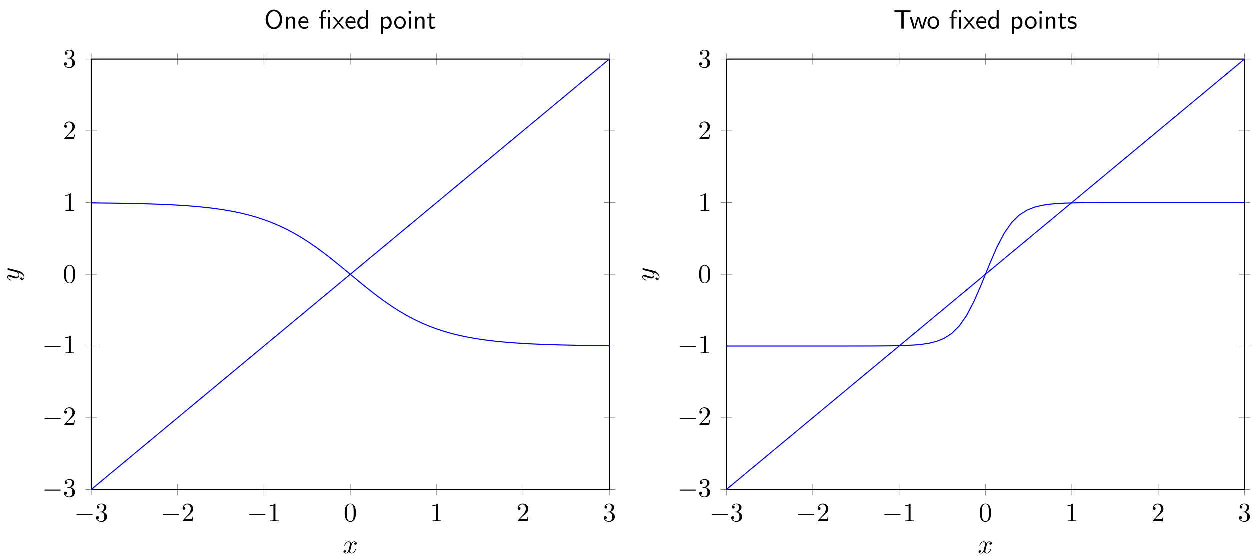
The bifurcation diagram is



3.4.9 We can try to solve for the fixed points like before, but we will get a transcendental equation. Instead, we can analyze this system graphically, since solving

$$0 = x + \tanh(rx)$$

is the same as finding the intersection between x and $-\tanh(rx)$. to show that there are two different types of behaviours:



The transition between one fixed point and two fixed point occurs when their slopes are the same at $x = 0$, i.e.

$$\left.\frac{d}{dx}\right|_{x=0} x = -\left.\frac{d}{dx}\right|_{x=0} \tanh(rx)$$
$$\implies 1 = -r\,\mathrm{sech}^2(0)$$
$$\implies r = -1.$$

At the $r_c = -1$ bifurcation, there is one fixed point, $x^* = 0$. Considering variations in x around x^*, we have

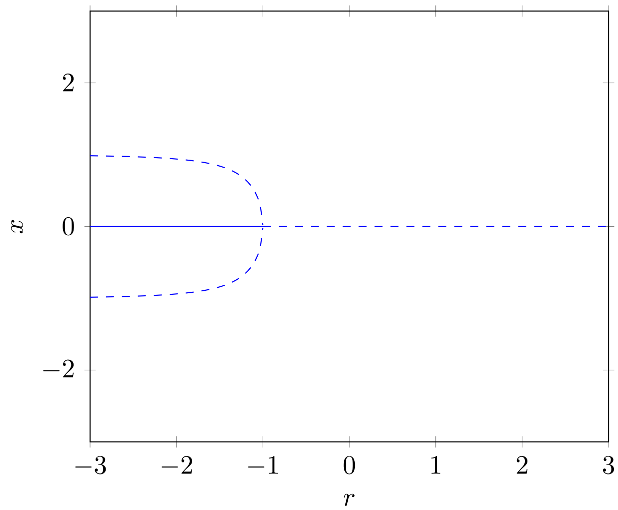
$$\dot{x} = x + rx - \frac{r^3x^3}{3} + \mathcal{O}(x^5)$$

Making the substitution $x \to \sqrt{3}X/r^2$ and $r \to R-1$, we have

$$\begin{aligned} \frac{\sqrt{3}\dot{X}}{r^{1.5}} &= \frac{\sqrt{3}X}{r^{1.5}} + \frac{\sqrt{3}X}{\sqrt{r}} - \frac{r^3 3\sqrt{3}X^3}{3r^{4.5}} \\ \implies \dot{X} &= X + rX - X^3 \\ \implies \dot{X} &= (1+r)X - X^3 \\ \implies \dot{X} &= RX - X^3, \end{aligned}$$

which is a super-critical pitchfork bifurcation.

Bifurcation Diagram of 3.4.9



3.4.10: We can solve for the fixed points. Similar to before, $x = 0$ is always a fixed point, so we will now find the fixed points for $0 = r + x^2/(1+x^2)$, which gives

$$\begin{aligned} 0 &= r + \frac{x^2}{1+x^2} \\ \implies r &= -\frac{x^2}{1+x^2} \\ \implies r &+ rx^2 + x^2 = 0 \\ \implies x^2&(r+1) + r = 0 \\ \implies x^2 &= -\frac{r}{r+1}. \end{aligned}$$

Note that x is only well-defined for $-1 < x \le 0$. We have the following cases:

- $r \le -1$: one solution at $x = 0$
- $-1 < r < 0$: one solution at $x = 0$ and two solutions at $x = \pm\sqrt{-r/(r+1)}$.
- $r \ge 0$: one solution at $x = 0$.

There are two bifurcations, one at $r = -1$ and one at $r = 0$. At $r = 0$, the fixed points approach $x^* = 0$ from both sides. Considering variations in x around x^*, we get

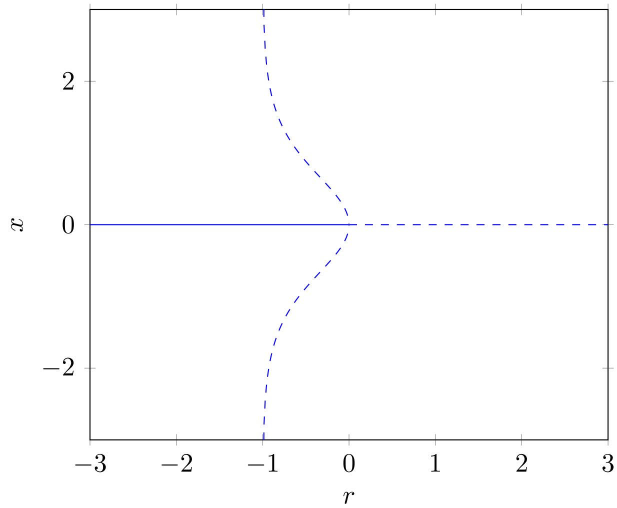
$$\begin{aligned} \dot{x} &= rx + x^3\left(1 - \mathcal{O}(x^2)\right) \\ &= rx + x^3 - \mathcal{O}(x^5). \end{aligned}$$

Ignoring the higher order terms, this gives us

$$\dot{x} = rx + x^3,$$

which we can recognize as the sub-critical pitchfork bifurcation. The other bifurcation at $r = -1$ is similar to 3.4.8, in the sense that when approaching $r_c = -1$ from the positive side, the fixed points diverge to $\pm\infty$, and then it doesn't make sense to talk about the normal form at some (r_c, x^*). The bifurcation diagram is

Bifurcation Diagram of 3.4.10



Problem Five

(i) We first start by finding a general formula for the power of the matrix.

Lemma 1: We have the following relationship

$$\begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & 0 \\ n\lambda^{n-1}c & \lambda^n. \end{pmatrix}$$

for $n \geq 0$.

Proof. We prove via induction. For the base case $n = 0$, we have

$$\begin{pmatrix} \lambda^0 & 0 \\ 0\lambda^{0-1}c & \lambda^0 \end{pmatrix} = I = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}^0.$$

Now suppose this is true for $n = k$. We will show that this is true for $n = k + 1$. We have

$$\begin{aligned} A^{k+1} &= A^k A \\ &= \begin{pmatrix} \lambda^k & 0 \\ k\lambda^{k-1}c & \lambda^k \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{k+1} & 0 \\ k\lambda^k c + \lambda^k c & \lambda^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{k+1} & 0 \\ (k+1)\lambda^k c & \lambda^{k+1} \end{pmatrix}, \end{aligned}$$

and we are done. □

We can compute e^{At} by computing the power series for each element. For the diagonal elements,

$$\sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} = e^{\lambda t},$$

since this is the standard expansion for the exponential. Next, we have a term that is more interesting,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n n \lambda^{n-1} c}{n!} &= 0 + \sum_{n=1}^{\infty} \frac{t^n n \lambda^{n-1} c}{n!} \\ &= ct \sum_{n=1}^{\infty} \frac{t^{n-1} \lambda^{n-1}}{(n-1)!} \\ &= ct \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{(n)!} \\ &= cte^{\lambda t}.\end{aligned}$$

Therefore,

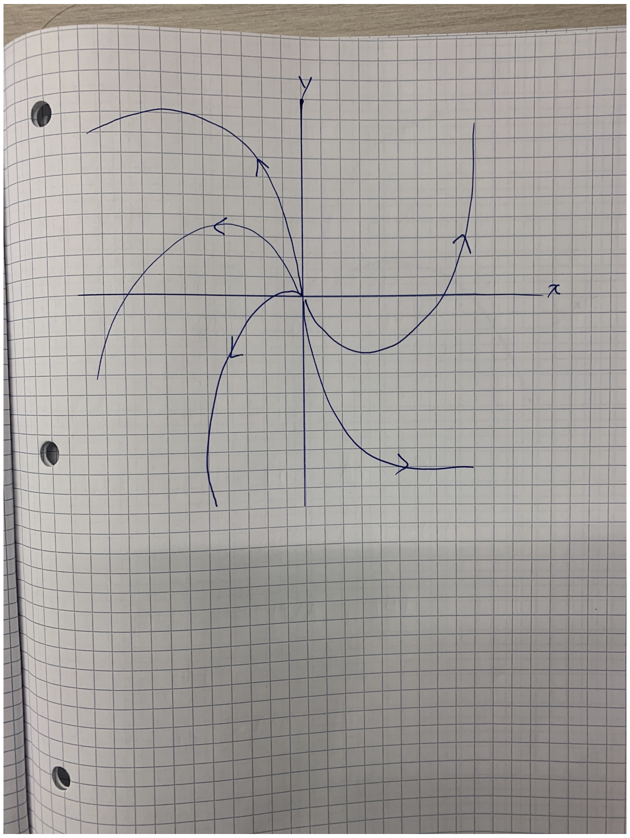
$$\begin{aligned}e^{At} &= \begin{pmatrix} e^{\lambda t} & 0 \\ cte^{\lambda t} & e^{\lambda t} \end{pmatrix} \\ &= e^{\lambda t} \begin{pmatrix} 1 & 0 \\ ct & 1 \end{pmatrix}.\end{aligned}$$

(ii) The solution for the flow is

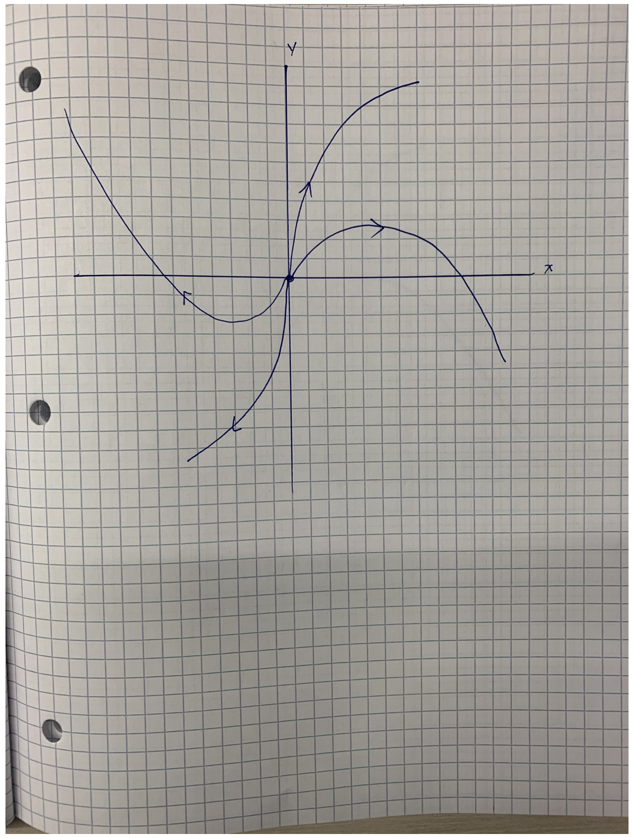
$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{\lambda t} \begin{pmatrix} 1 & 0 \\ ct & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \\ &= e^{\lambda t} \begin{pmatrix} x(0) \\ x(0)ct + y(0). \end{pmatrix}\end{aligned}$$

Let us look at some specific cases. In all these cases, we have a repeated eigenvalue of λ, which is real, so we'll have **degenerate nodes**.

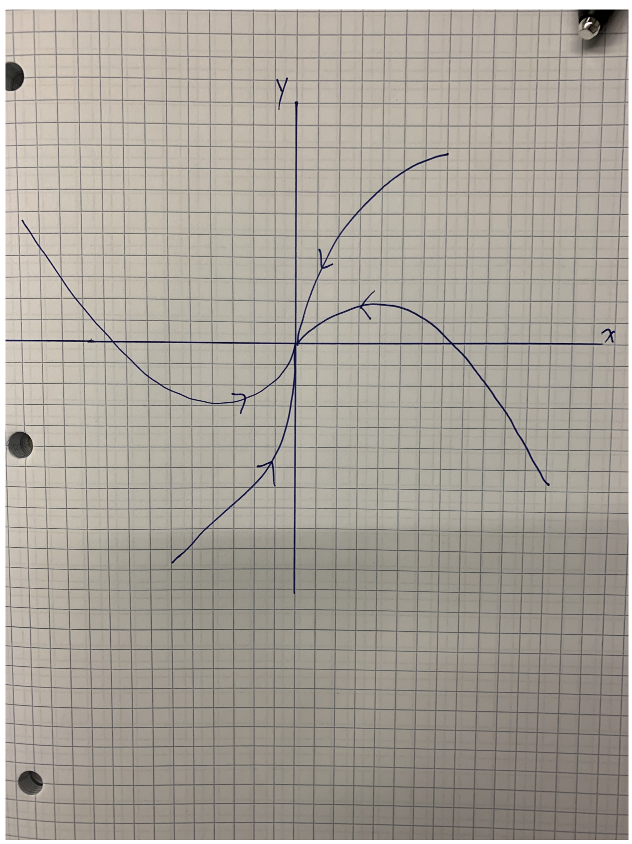
- $\lambda = 2; c = 3$: Since $\lambda > 0$, we have an unstable fixed point.



- $\lambda = 2; c = -3$: Since $\lambda > 0$, we have an unstable fixed point.



- $\lambda = -2, c = 2$: Since $\lambda < 0$, we have an attracting stable fixed point.



Problem Six

5.2.4 We have

$$\dot{x} = 5x + 10y$$
$$\dot{y} = -x - y$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

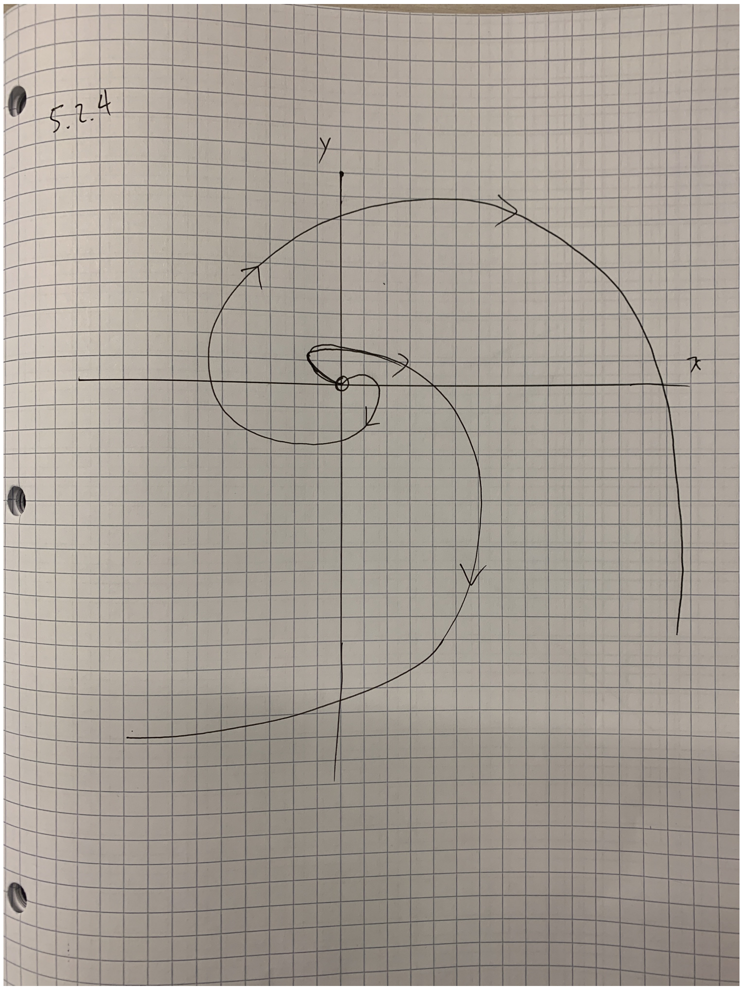
The eigenvalues are given by

$$(5-\lambda)(-1-\lambda) + 10 = 0 \iff \lambda^2 - 4\lambda + 5 = 0,$$

which has solutions in the form of

$$\lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i.$$

Since these are complex eigenvalues with a nonzero real component, the fixed point will be an **unstable spiral.** The stability and direction of flow was determined by determining the vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at a few places.



5.2.5 We have

$$\dot{x} = 3x - 4y$$
$$\dot{y} = x - y,$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

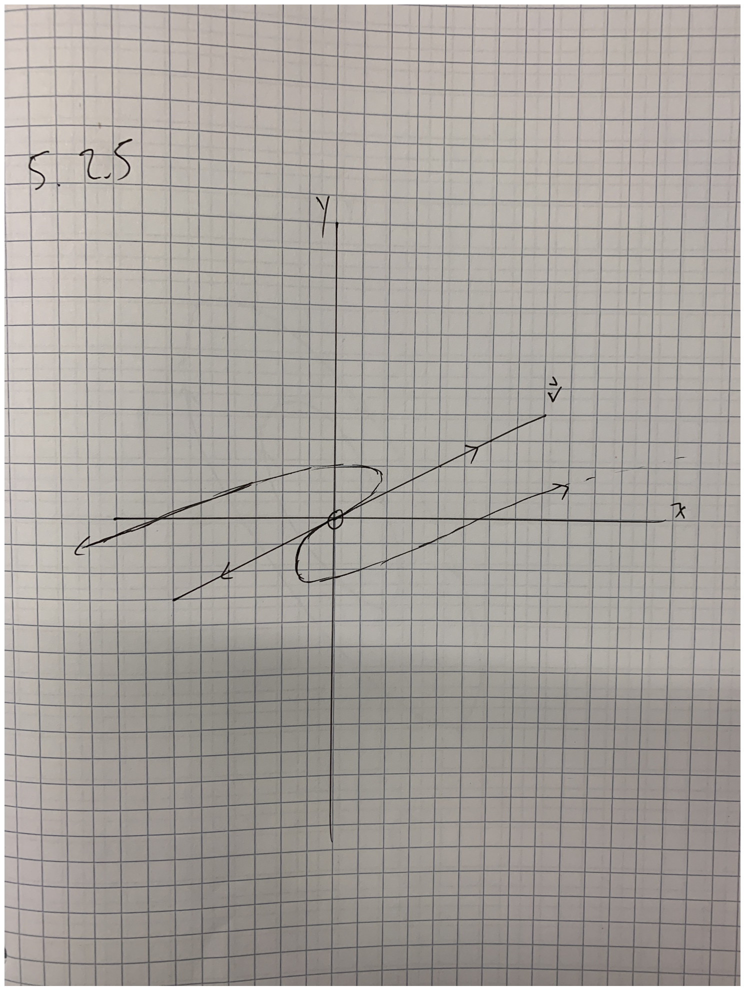
This has eigenvalues given by

$$(3-\lambda)(-1-\lambda)+4=0 \iff (\lambda-1)^2=0,$$

which has a repeating eigenvalue of $\lambda = 1$. Let us determine the eigenspace. We have

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

The rank is 1, so the eigenspace is given by the span of a single vector, $\left\{ t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$, so the fixed point is an **unstable degenerate node**. In the below diagram, the straight line corresponds to the eigenvector $\vec{v}$. Again, the stability and direction of flow was determined by determining the vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at a few important places (i.e. on the eigenvector and on the x-axis).



5.2.7 We have

$$\dot{x} = 5x + 2y$$
$$\dot{y} = -17x - 5y$$

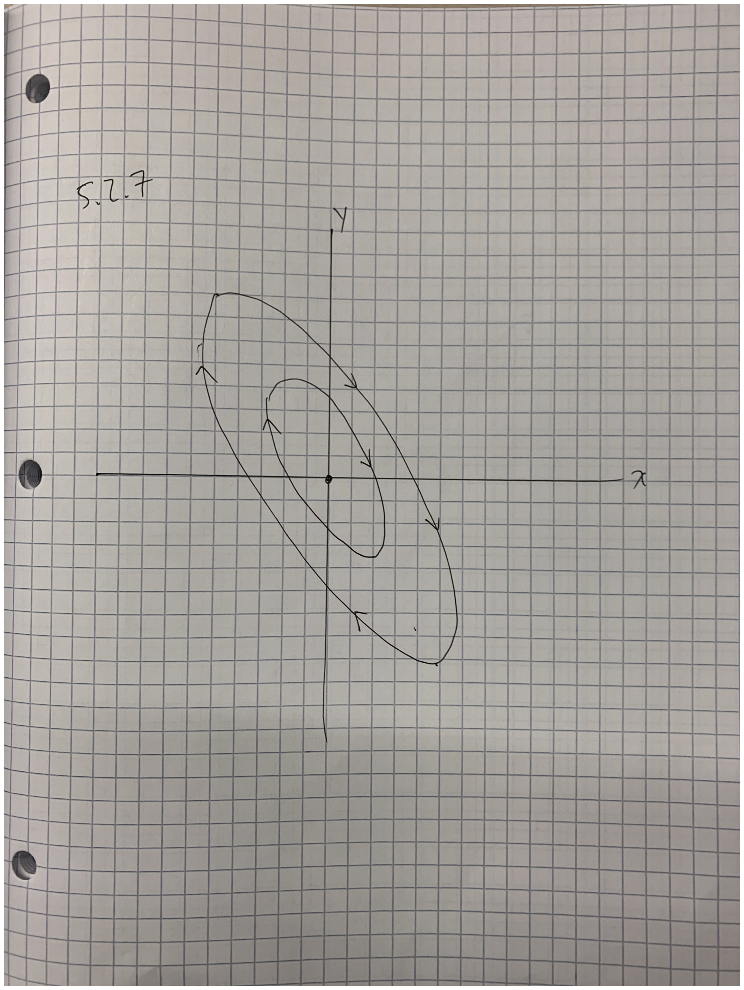
and the matrix form is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which has eigenvalues given by

$$(5-\lambda)(-5-\lambda)+34=0 \iff \lambda^2+9=0,$$

which has eigenvalues $\lambda = \pm 3i$. Since they are complex eigenvalues with no real component, the fixed point is a **neutrally stable center**. The direction of flow was determined by determining the vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at a few places (i.e. on the x-axis and on the y-axis).



5.2.10 We have

$$\dot{x} = y$$
$$\dot{y} = -x + 2y,$$

and the matrix form is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

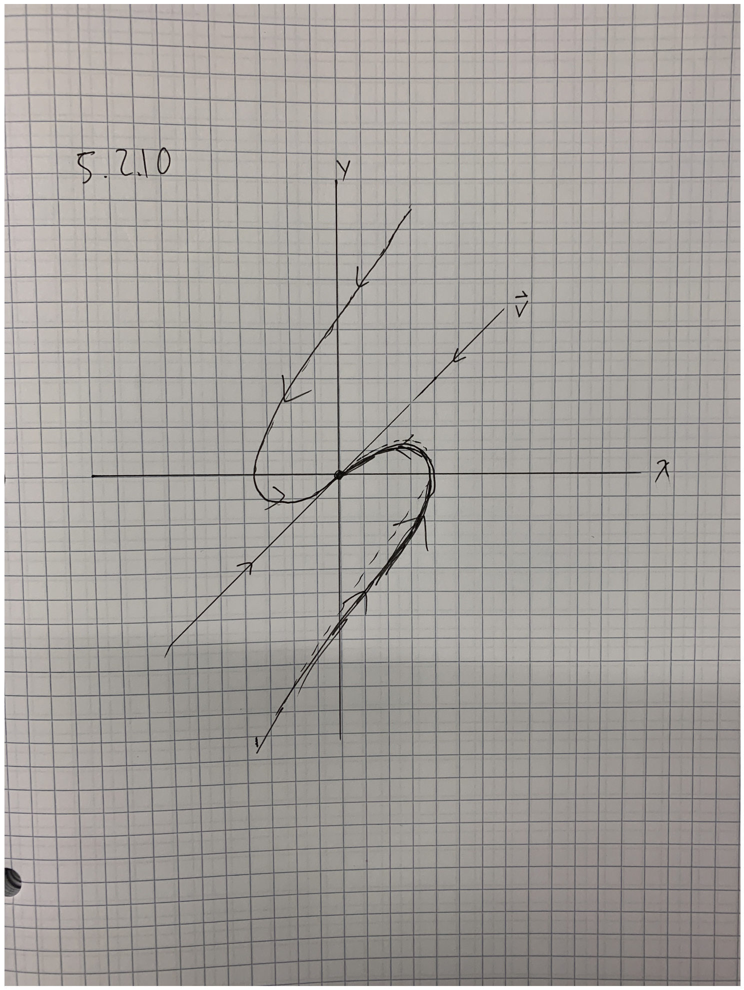
The eigenvalues are given by

$$(0 - \lambda)(2 - \lambda) + 1 = 0 \iff \lambda^2 - 2\lambda + 2 = 0 \iff (\lambda - 1)^2 = 0.$$

This gives the repeated eigenvalue of $\lambda = 1$. Let us determine the eigenspace. We have

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This matrix has rank 1 and so the eigenspace is given by the span of a single vector, $\left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$, so the fixed point is a **stable degenerate node**. The straight line corresponds to the eigenvector $\vec{v}$. The direction and stability of flow was determined by determining the vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at a few places (i.e. on the eigenvector and on the x-axis).



5.2.10
y
$\vec{v}$
x