PHY460: Nonlinear Physics
Problem Set 3

QiLin Xue

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Problem One

6.3.11

(a) The system is given by

$$\dot{r} = -r \tag{0.1}$$
$$\dot{\theta} = \frac{1}{\ln r}. \tag{0.2}$$

Solving the first equation gives $r = r_0 e^{-t}$ and plugging this into the second equation and solving gives

$$\dot{\theta} = \frac{1}{\ln(r_0) + \ln(e^{-t})} \tag{0.3}$$
$$\implies \int \mathrm{d}\theta = \int \frac{1}{\ln(r_0) - t}\,\mathrm{d}t \tag{0.4}$$
$$\implies \theta(t) = \theta_0 - \ln\left|1 - \frac{t}{\ln r_0}\right| \tag{0.5}$$

(b) As $t \to \infty$, clearly $r \to 0$. For $\theta(t)$, note that as $t \to \infty$, we have $\dfrac{t}{\ln r_0} > 1$, so

$$\theta(t) = \theta_0 - \ln\left(\frac{t}{\ln r_0} - 1\right). \tag{0.6}$$

The logarithm term approaches ∞ as $t \to \infty$ and therefore $\theta(t)$ approaches $-\infty$ as $t \to \infty$. Therefore, $|\theta(t)| \to \infty$.

(c) Let $x = r\cos\theta$ and $y = r\sin\theta$, such that

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \tag{0.7}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}. \tag{0.8}$$

Substituting in expressions for $\dot{r}, \dot{\theta}$, we get

$$\dot{x} = -r\cos\theta - \frac{r}{\ln r}\sin\theta \tag{0.9}$$
$$\dot{y} = -r\sin\theta + \frac{1}{\ln r}\cos\theta, \tag{0.10}$$

and changing everything back to cartesian coordinates again gives

$$\dot{x} = -x - \frac{1}{2\ln(x^2+y^2)}y \tag{0.11}$$
$$\dot{y} = -y + \frac{1}{2\ln(x^2+y^2)}x. \tag{0.12}$$

(d) Note that at the fixed point $(x, y) = (0, 0)$, we have that $\ln(x^2 + y^2) \to \infty$ (really fast) so $\frac{1}{\ln(x^2+y^2)} \to 0$. Therefore, after linearizing it, we should expect

$$\dot{x} = -x \tag{0.13}$$
$$\dot{y} = -y. \tag{0.14}$$

But we can make this argument more rigorous. The Jacobian can be computed to be

$$\mathcal{J} = \begin{pmatrix} -1 - \dfrac{yx}{(y^2+x^2)\ln^2(x^2+y^2)} & \dfrac{1}{2\ln(x^2+y^2)} - \dfrac{y^2}{(y^2+x^2)\ln^2(x^2+y^2)} \\ \dfrac{1}{2\ln(x^2+y^2)} - \dfrac{x^2}{(y^2+x^2)\ln^2(x^2+y^2)} & -1 - \dfrac{xy}{(y^2+x^2)\ln^2(x^2+y^2)} \end{pmatrix}. \tag{0.15}$$

At (x, y) this becomes

$$\mathcal{J}|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{0.16}$$

which agrees with what we postulated earlier about the linear system. Therefore, the linear system is a stable star.

6.3.13

We have the system

$$\dot{x} = -y - x^3 \tag{0.17}$$
$$\dot{y} = x. \tag{0.18}$$

We can convert this to polar coordinates. Note that $r = \sqrt{x^2 + y^2}$, so taking the time derivative yields

$$\dot{r} = \frac{1}{\sqrt{x^2+y^2}}(x\dot{x} + y\dot{y}) \tag{0.19}$$
$$= \frac{1}{r}\left(-yx - x^4 + xy\right) \tag{0.20}$$
$$= -\frac{x^4}{r} \tag{0.21}$$
$$= -r^3\cos^3\theta. \tag{0.22}$$

Similarly we have $\theta = \tan^{-1}(y/x)$, so its time derivative yields

$$\dot{\theta} = \frac{1}{1+(y/x)^2} \cdot \frac{\dot{y}x - y\dot{x}}{x^2} \tag{0.23}$$
$$= \frac{\dot{y}x - y\dot{x}}{x^2 + y^2} \tag{0.24}$$
$$= \frac{x^2 + y^2 + yx^3}{x^2 + y^2} \tag{0.25}$$
$$= \frac{r^2 + r^4 \sin\theta\cos^3\theta}{r^2} \tag{0.26}$$
$$= 1 + r^2\sin\theta\cos^3\theta. \tag{0.27}$$

So the system now becomes

$$\dot{r} = -r^3\cos^3\theta \tag{0.28}$$
$$\dot{\theta} = 1 + r^2\sin\theta\cos^3\theta. \tag{0.29}$$

Note that for $r < 1$ we have that $\dot{\theta} > 0$ since $\sin\theta\cos^3\theta$ is bounded below -1 (though it is not a tight lower bound). Moreover, $\dot{\theta}$ is bounded below by some nonzero value, so $\theta(t) \to \infty$ as $t \to \infty$.

Second, we will show that $(0, 0)$ is a local minimum, and we shall do this by using the Liapunov function

$$V(x, y) = x^2 + y^2. \tag{0.30}$$

This is clearly nonzero everywhere except for $(0,0)$, and its time derivative

$$\dot{V}(x,y) = 2x\dot{x} + 2y\dot{y} \tag{0.31}$$
$$= -2x^4, \tag{0.32}$$

is always negative for $x \neq 0$. Note that $(0,y)$ could also result in $\dot{V}(0,y) = 0$, but in that case $\dot{\theta} = 1$, moving us away from the $x = 0$ line, and allowing energy to decrease again. Therefore, we have $r \to 0$ as $t \to \infty$ and this is a stable spiral.

In terms of a linear analysis, linearizing around $(0,0)$ gives

$$\mathcal{J}|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

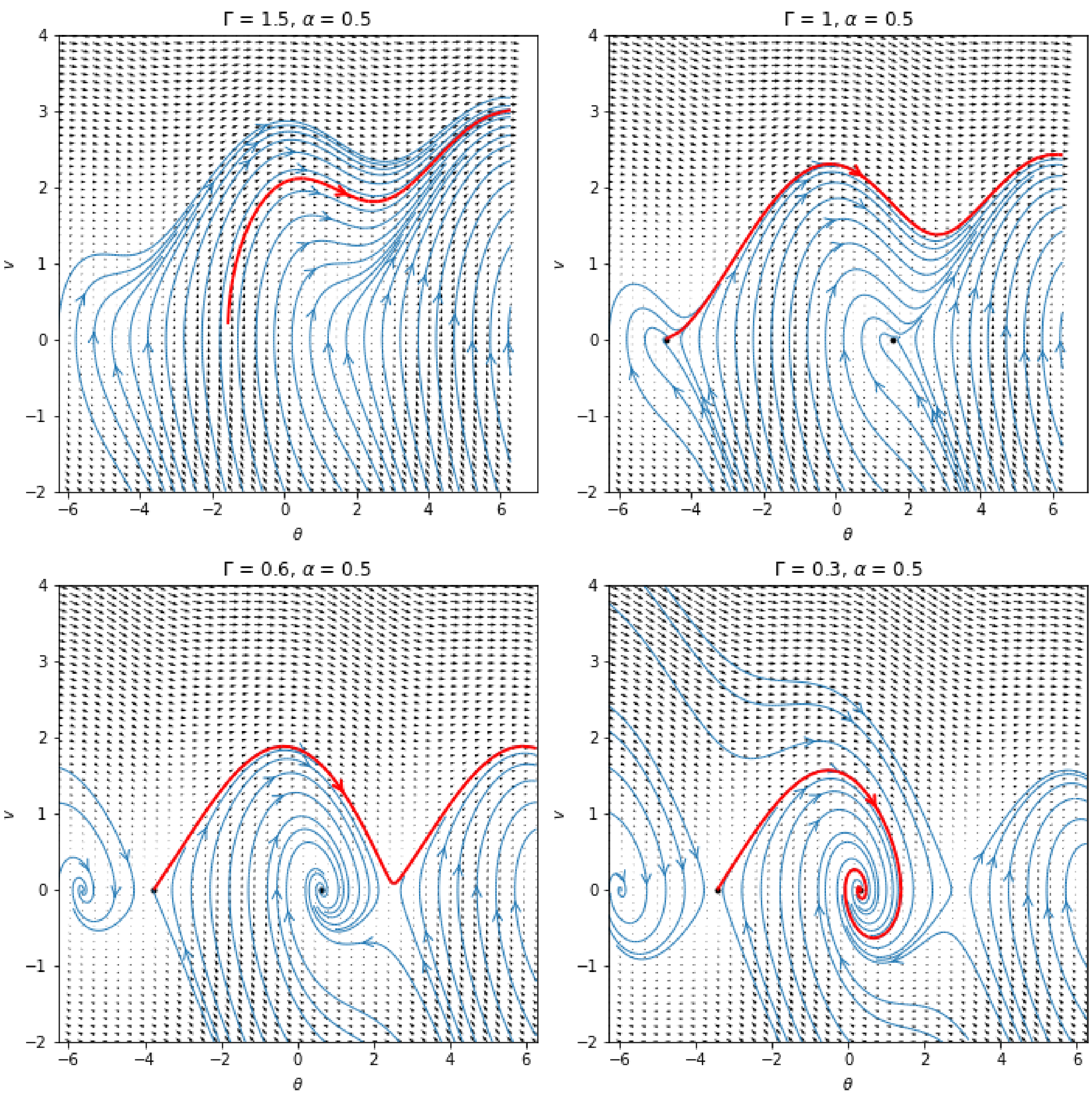
The trace is 0 and the determinant is 1, so we have a center.

Problem Three

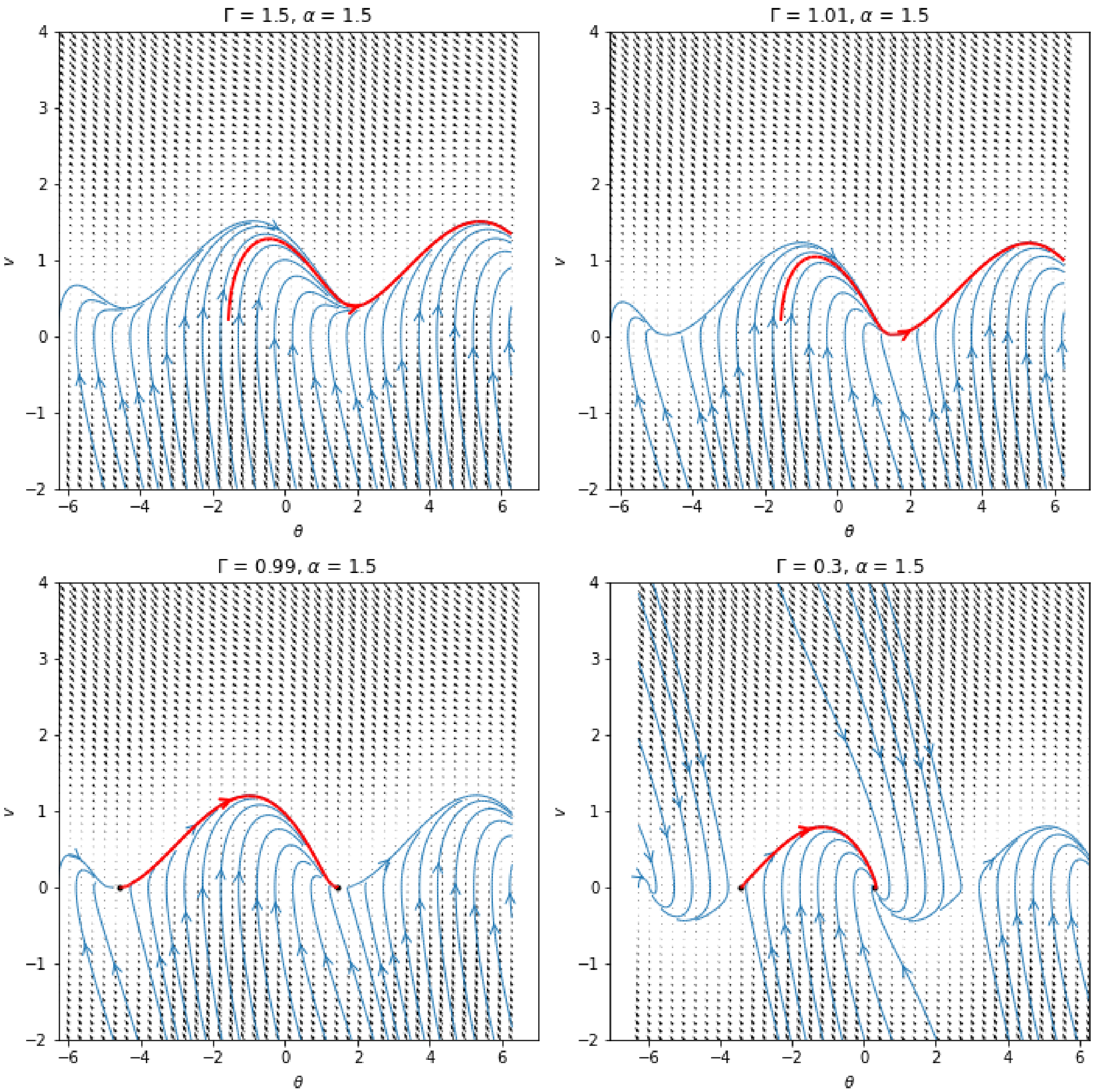
(i) We can make the substitution $u = x, v = \dot{x}$ such that the differential equation is in the form of

$$\dot{u} = v \tag{0.33}$$
$$\dot{v} = \Gamma - \alpha v - \sin u. \tag{0.34}$$

For sufficiently small α, i.e. $\alpha = 0.5$, we can numerically compute the phase portrait to be the below.



We see that a homoclinic bifurcation occurs somewhere between $\Gamma = 0.6$ and $\Gamma = 0.3$. If α is very large, i.e. $\alpha = 1.5$, we have an infinite-period bifurcation at $\Gamma = 1$.



(ii) We want $\ddot{x}$ and $\dot{x}$ to both be zero. This gives

$$\sin x = \Gamma,$$

so $x^* = \sin^{-1}\Gamma, \pi - \sin^{-1}\Gamma$ with $\dot{x}^* = 0$ for $\Gamma \leq 1$. For $\Gamma > 1$, there are no fixed points. Consider the $\Gamma < 1$ case. We can make the same substitution as part (a) to get

$$\dot{u} = v \tag{0.35}$$
$$\dot{v} = \Gamma - \alpha v|v| - \sin u. \tag{0.36}$$

The Jacobian is

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -\cos u & -2\alpha|v| \end{pmatrix}, \tag{0.37}$$

where tr$(\mathcal{J}) = 0$ and det$(\mathcal{J}) = \cos\left(\sin^{-1}\Gamma\right), \cos\left(\pi - \sin^{-1}\Gamma\right)$ for the two fixed points. This can be rewritten as

$$\det(\mathcal{J})_1 = \sqrt{1-\Gamma^2} > 0 \tag{0.38}$$
$$\det(\mathcal{J})_2 = -\sqrt{1-\Gamma^2} < 0. \tag{0.39}$$

Therefore, the fixed point at $\pi - \sin^{-1}\Gamma$ is a saddle point, and the fixed point at $\sin^{-1}\Gamma$ is a center node, according to linearization.

To determine if the latter fixed point is truly a center node, we can consider the Liapunov function

$$V(u,v) = -\Gamma u + \frac{1}{2}v^2 - \cos u - \left(-\Gamma \sin^{-1}\Gamma - \sqrt{1-\Gamma^2}\right)$$

We first need to show that there exists an open neighborhood around $(u^*, v^*) = (\sin^{-1}\Gamma, 0)$ such that $V(u,v) > 0$ for all $(u,v) \neq (u^*, v^*)$. Consider $(u,v) = (u^* + \delta, v^* + \epsilon)$, where $\delta, \epsilon > 0$. Then,

$$V(u,v) = -\Gamma u^* - \Gamma\delta + \frac{1}{2}((v^*)^2 + 2v^*\epsilon + \epsilon^2) - \cos(u^* + \delta) - \left(-\Gamma \sin^{-1}\Gamma - \sqrt{1-\Gamma^2}\right) \tag{0.40}$$

$$= \left(-\Gamma u^* + \frac{1}{2}(v^*)^2 - \cos(u^*)\right) - \Gamma\delta + v^*\epsilon + \frac{1}{2}\epsilon^2 + \delta\sin(u^*) - \left(-\Gamma \sin^{-1}\Gamma - \sqrt{1-\Gamma^2}\right) \tag{0.41}$$

$$= \left(-\Gamma \sin^{-1}\Gamma - \sqrt{1-\Gamma^2}\right) - \left(-\Gamma \sin^{-1}\Gamma - \sqrt{1-\Gamma^2}\right) - \Gamma\delta + v^*\epsilon + \frac{1}{2}\epsilon^2 + \delta\sin(u^*) \tag{0.42}$$

$$= \frac{1}{2}\epsilon^2. \tag{0.43}$$

There then exists an open neighborhood around (u^*, v^*) such that for any choice of δ, ϵ, the above quantity will be nonzero. Next, we need to show that $\dot{V}(u,v) < 0$ for all $(u,v) \neq (u^*, v^*)$. This is the easy part, and what motivated the construction of this function in the first place. Note that,

$$\dot{V}(u,v) = -\Gamma\dot{u} + v\dot{v} + \dot{u}\sin u \tag{0.44}$$

$$= v\left(-\Gamma + \dot{v} + \sin u\right) \tag{0.45}$$

$$= v\left(-\Gamma + \Gamma - \alpha v|v| - \sin u + \sin u\right) \tag{0.46}$$

$$= \alpha v^2|v| < 0. \tag{0.47}$$

Thus, the actual behavior is a stable spiral.

(iii) Consider a trajectory that starts at $u = 0$, $v = v_1$, and define the Poincare map to be the value of v of the solution curve at $u = 2\pi$. We wish to show that there exists a v_0 such that $v_0 = P(v_0)$.

Consider the nullcline

$$v = \sqrt{\frac{\Gamma - \sin u}{\alpha}} \tag{0.48}$$

for $\dot{v} = 0$. Note that the absolute value signs were taken care of since we know that $\Gamma > \sin u$, so v does not have any switch in signs, so v is always positive. Note that above this nullcline (increasing v), we will have flow that points downwards and below this nullcline we will have flow that points upwards.

This nullcline is also bounded above by some v_2 and bounded below by some v_1, and since the flows point towards the nullcline, the region $v_1 \leq v \leq v_2$ is a trapping region. We've shown that,

(a) $R = S^1 \times [v_1, v_2]$ is a closed bounded 2-dimensional region.

(b) $(\dot{u}, \dot{v})$ is continuously differentiable in R (this is true since $\dot{v}$ doesn't change direction, so no weird things happen there)

(c) R does not contain fixed points (shown in part (ii))

(d) There exists a trajectory confined in R (true since we know that R is a trapping region)

According to the Poincaré-Bendixson Theorem, there exists a closed orbit inside R.

To show that there exists a limit cycle, and moreover a unique limit cycle, recall that we only need to consider orbits that are topologically the same as rotations, since the only other type of orbit, librations, requires there to be a fixed point, which we do not have.

Consider the energy function

$$E = \frac{1}{2}v^2 - \cos u \tag{0.49}$$

as before. We have

$$\frac{dE}{du} = v\frac{dv}{du} + \sin u \tag{0.50}$$

and

$$\frac{dv}{du} = \frac{\dot{v}}{\dot{u}} = \frac{\Gamma - \alpha v|v| - \sin u}{v}, \tag{0.51}$$

so that

$$\begin{aligned}\frac{dE}{du} &= v\left(\frac{\Gamma - \alpha v|v| - \sin u}{v}\right) + \sin u &(0.52)\\ &= \Gamma - \alpha v|v|. &(0.53)\end{aligned}$$

Energy conservation tells us that the change in energy after one rotation should be zero, i.e.

$$\int_0^{2\pi} \frac{dE}{du}\,\mathrm{d}u = \int_0^{2\pi} (\Gamma - \alpha v|v|)\,\mathrm{d}u = 0, \tag{0.54}$$

which is equivalent to

$$\int_0^{2\pi} \Gamma\,\mathrm{d}u = \alpha \int_0^{2\pi} v^2\,\mathrm{d}u\,. \tag{0.55}$$

Here, we used the fact that $|v| = v$, which is subtle. It relies on the fact that the choice for the boundary of our trapping region can be picked such that v is always positive inside, i.e. $v_1 > 0$ and v_2 can be arbitrarily big (since the nullcline doesn't cross $v = 0$). However, this argument will only show that this is a unique limit cycle for $v > 0$. Note that closed orbits cannot exist for $v < 0$ since $\dot{v} > 0$, so it cannot be periodic.

Continuing with our previous argument, we have that

$$\int_0^{2\pi} v^2\,\mathrm{d}u = \frac{2\pi\Gamma}{\alpha}. \tag{0.56}$$

Now consider two closed orbits described by v and $\tilde{v}$ and suppose without loss of generality that $v > \tilde{v}$ (which is possible since trajectories can't cross). Then

$$\int_0^{2\pi} v^2\,\mathrm{d}u > \int_0^{2\pi} \tilde{v}^2\,\mathrm{d}u\,,$$

giving us a contradiction. Therefore, there is only one closed orbit, which is our unique limit cycle.

(iv) Since we've already defined u, we will rename the new dependent varible as $\chi = \frac{1}{2}\dot{x}^2$. Then

$$\begin{aligned}\frac{d\chi}{du} &= \frac{1}{2}\frac{d}{du}v^2 &(0.57)\\ &= v\frac{dv}{du} &(0.58)\\ &= \Gamma - \alpha v|v| - \sin u &(0.59)\\ &= \dot{v} = \ddot{x}, &(0.60)\end{aligned}$$

where we used our earlier computation of $\frac{dv}{du}$ in line 3.

(v) We are now able to write out differential equation in the form of

$$\frac{d\chi}{dx} + \underbrace{2\alpha}_{P(x)}\chi = \underbrace{\Gamma - \sin x}_{Q(x)}. \tag{0.61}$$

This is a standard first order differential equation where the solution can be obtained using the integrating factor method. The integrating factor is

$$I(x) = \exp\left(\int P(x)\,\mathrm{d}x\right) = e^{2\alpha x},$$

and the general solution is

$$\begin{aligned}\chi &= \frac{1}{I(x)}\left[\int I(x)Q(x)\,\mathrm{d}x + C\right] \\ &= e^{-2\alpha x}\left[\int e^{2\alpha x}\left(\Gamma - \sin x\right)\mathrm{d}x + C\right] \\ &= \frac{\Gamma}{2\alpha} - \frac{2\alpha\sin(x) - \cos(x)}{4\alpha^2+1} + Ce^{-2\alpha x},\end{aligned}$$

where the constant C allows us to get all trajectories. To only get the limit cycle, we need to make $\dot{x}$ periodic, which can only be done by setting $C = 0$. Solving for $\dot{x}$ gives us,

$$\dot{x} = \sqrt{\frac{\Gamma}{\alpha} - \frac{4\alpha\sin(x) - 2\cos(x)}{4\alpha^2+1}}.$$

(vi) The above equation can be written in a more suggestive manner by noting that

$$A\sin(x) + B\cos(x) = \sqrt{A^2+B^2}\sin(x+\theta), \tag{0.62}$$

where $\theta = \tan^{-1}\left(\frac{B}{A}\right)$. Then, we have

$$\dot{x} = \sqrt{\frac{\Gamma}{\alpha} - \frac{2\sin(x+\theta)}{\sqrt{4\alpha^2+1}}}.$$

The explicit formula can be determined by finding the value of Γ such that the value inside the square root could be zero (i.e. when the sine value is 1). This is given by

$$\Gamma_c = \frac{2\alpha}{\sqrt{4\alpha^2+1}}.$$

Now, we need to argue that this is a homoclinic bifurcation. To do so, we need to show that the orbit moves closer and closer to a saddle point, which we already classified to be at $x = \pi - \sin^{-1}\Gamma$. Note that $\Gamma_c < 1$, so that in the range $\Gamma_c < \Gamma < 1$, we have both a periodic cycle, as well as two fixed points: a spiral node and a saddle node. I'm not sure how to rigorously show that it approaches the saddle node.

Problem Four

(a) For $V = x^2 + y^2$, we have the system

$$\dot{x} = 2x \tag{0.63}$$
$$\dot{y} = 2y. \tag{0.64}$$

(b) For $V = x^2 - y^2$, we have the system

$$\dot{x} = 2x \tag{0.65}$$
$$\dot{y} = -2y. \tag{0.66}$$

(c) For $V = e^x \sin y$, we have the system

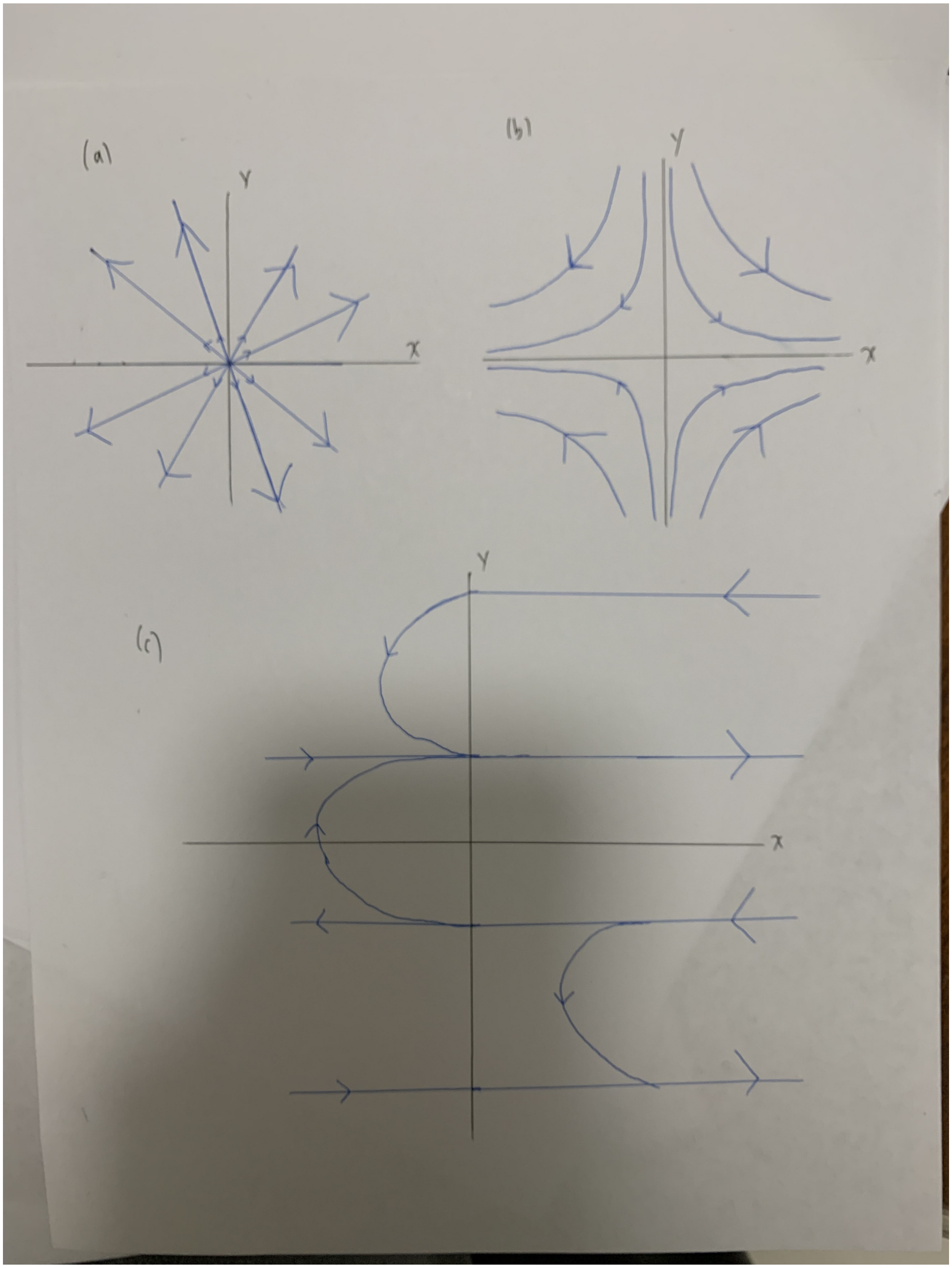
$$\dot{x} = e^x \sin y \tag{0.67}$$
$$\dot{y} = e^x \cos y. \tag{0.68}$$

This is a tricky system that is very hard to visualize. However, note that at a given x value, we have the simple 1d equation

$$\dot{y} = A\cos y, \tag{0.69}$$

where A is some positive constant.

The hand sketches are attached the following page.


(a)
Y
x
(b)
Y
x
(c)
Y
x

Problem Five

We have the equation

$$\ddot{x} + b\dot{x} - kx + x^3 = 0. \tag{0.70}$$

Consider the substitution $u = x$ and $v = \dot{x}$. Then, we have,

$$\dot{u} = v \tag{0.71}$$

$$\dot{v} = -bv + ku - u^3. \tag{0.72}$$

The Jacobian is

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ k - 3u^2 & -b. \end{pmatrix} \tag{0.73}$$

We have

$$\Delta = 3u^2 - k \tag{0.74}$$

$$\tau = -b, \tag{0.75}$$

so

$$\tau^2 - 4\Delta = b^2 + 4k - 12u^2. \tag{0.76}$$

For this system to have a fixed point, we need $v = 0$ and

$$ku - u^3 = 0 \iff u(\sqrt{k} - u)(\sqrt{k} + u) = 0. \tag{0.77}$$

For $u = 0$, we have the bifurcation curve

$$b^2 + 4k = 0, \tag{0.78}$$

and for $u = \pm\sqrt{k}$, we have the bifurcation curve

$$b^2 - 8k = 0. \tag{0.79}$$

We now consider the stability of each fixed point in each region. Note that for all the fixed points, $b > 0$ is stable and $b < 0$ is unstable. For the fixed point corresponding to $u^* = 0$, we have:

$$\Delta = -k \tag{0.80}$$

$$\tau = -b \tag{0.81}$$

so we have the standard Poincaré diagram but with the axes inverted. Note that when $4k = -b^2$, the Jacobian becomes

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -\dfrac{b^2}{4} & -b. \end{pmatrix} \tag{0.82}$$

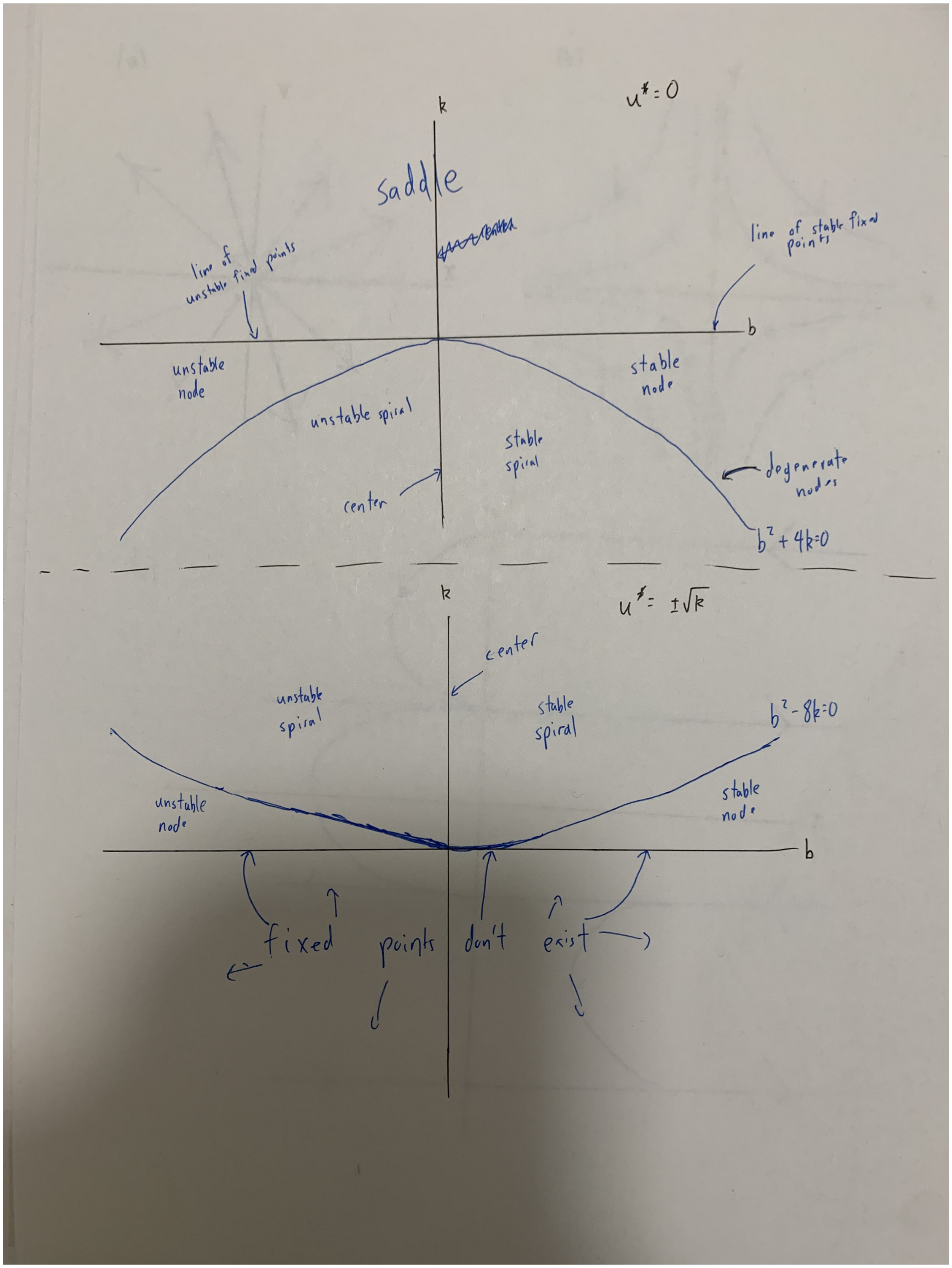
Clearly, not every vector is an eigenvector, so this corresponds to a degenerate node.

For the fixed point corresponding to $u^* = \sqrt{k}$, we have

$$\Delta = 2k \tag{0.83}$$

$$\tau = -b. \tag{0.84}$$

Note that this fixed point only exists for $k \geq 0$, so we have a scaled version of the standard Poincaré diagram but the lower half corresponds to no stable point. Note that the bifurcation curve corresponds to a degenerate node instead of a star node, per the same line of reasoning as before. The plots are shown below,



$u^* = 0$
k
b
saddle
line of unstable fixed points
line of stable fixed points
unstable node
unstable spiral
stable spiral
stable node
center
degenerate nodes
$b^2 + 4k = 0$
$u^* = \pm\sqrt{k}$
k
b
center
unstable spiral
stable spiral
$b^2 - 8k = 0$
unstable node
stable node
fixed points don't exist