# PHY460: Nonlinear Physics
# Problem Set 4

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## Problem One

(a) Recall from section 9.2 in the textbook that the rate of change of volume in phase space (or in this case, area) is given by

$$\dot{A} = \int_A \nabla \cdot \boldsymbol{f} \, \mathrm{d}A \,, \tag{0.1}$$

where

$$\nabla \cdot \boldsymbol{f} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = -2xy + 2xy = 0. \tag{0.2}$$

Therefore, $\frac{dA}{dt} = 0$ so the rate of change of phase space area is zero.

(b) Consider $x = r\cos\theta$ and $y = r\sin\theta$, such that $x^2 + y^2 = r^2$. Note that we've shown in the previous problem set an expression for $\dot{r}$,

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} \tag{0.3}$$

$$= \frac{-xy(1 + x^2 + y^2) + xy(1 + x^2 + y^2)}{r} \tag{0.4}$$

$$= 0, \tag{0.5}$$

and

$$\dot{\theta} = \frac{\dot{y}x - y\dot{x}}{r^2} \tag{0.6}$$

$$= \frac{x^2(1 + x^2 + y^2) + y^2(1 + x^2 + y^2)}{r^2} \tag{0.7}$$

$$= (\cos^2\theta + sin^2\theta)(1 + r^2) \tag{0.8}$$

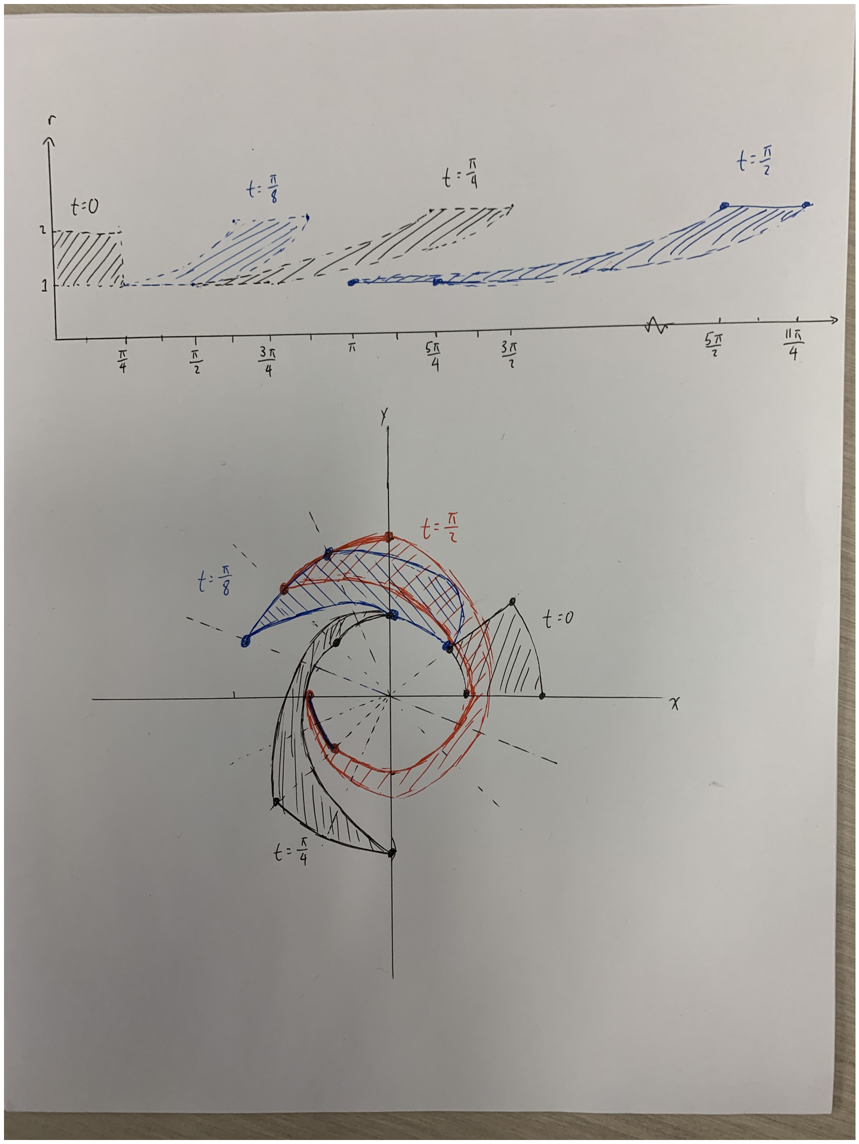
$$= 1 + r^2. \tag{0.9}$$

(c) Using $\dot{r} = 0$ and $\dot{\theta} = 1 + r^2$, we can solve:

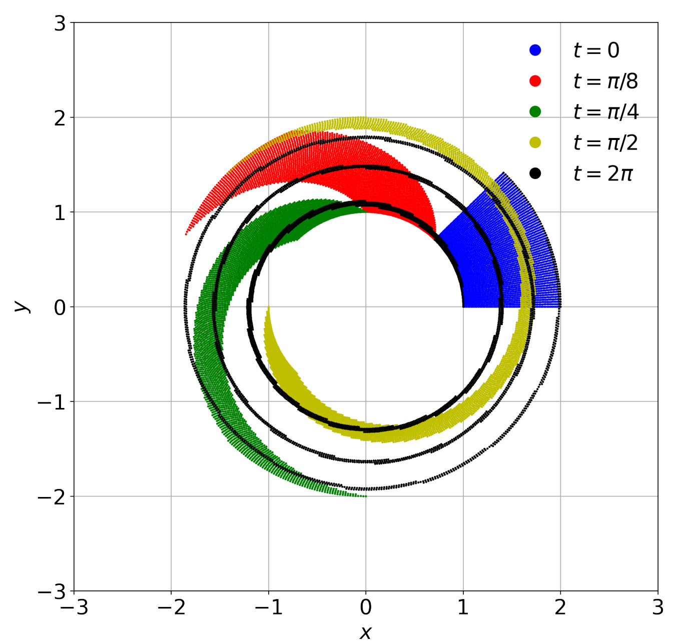
$$r(t) = r_0 \tag{0.10}$$

$$\theta(t) = \theta_0 + (1 + r_0^2)t, \tag{0.11}$$

where $(r_0, \theta_0)$ are the initial conditions. We can make a plot in both polar and Cartesian space, the trajectory of the particles that originally formed a square in polar space.




(d) See the above plot. Because the plot in Cartesian coordinates can be confusing, I've made a clearer diagram with Python,

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Note that as $t \to \infty$, the region becomes stretched into an extremely thin spiral that ranges from $r = 1$ to $r = 2$.

(e) We now have,

$$\dot{x} = -y(1 + x^2 + y^2) + x(1 - x^2 - y^2) \tag{0.12}$$
$$\dot{y} = x(1 + x^2 + y^2) + y(1 - x^2 - y^2). \tag{0.13}$$

We can compute,

$$\begin{aligned} \nabla \cdot \boldsymbol{f} &= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} &(0.14)\\ &= \left(-2xy + 1 - 3x^2 - y^2\right) + \left(2xy + 1 - x^2 - 3y^2\right) &(0.15)\\ &= 2 - 4(x^2 + y^2). &(0.16)\end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{A}\int_A \nabla \cdot \boldsymbol{f} \,\mathrm{d}A &(0.17)\\ &\qquad = \frac{1}{A}\left(A \cdot \langle \nabla \cdot f\rangle\right) &(0.18)\\ &\qquad = \langle 2 - 4(x^2 + y^2)\rangle &(0.19)\\ &\qquad = 2 - 4\langle x^2 + y^2\rangle, &(0.20)\end{aligned}$$

or in polar coordinates, becomes

$$2 - 4\langle r\rangle. \tag{0.21}$$

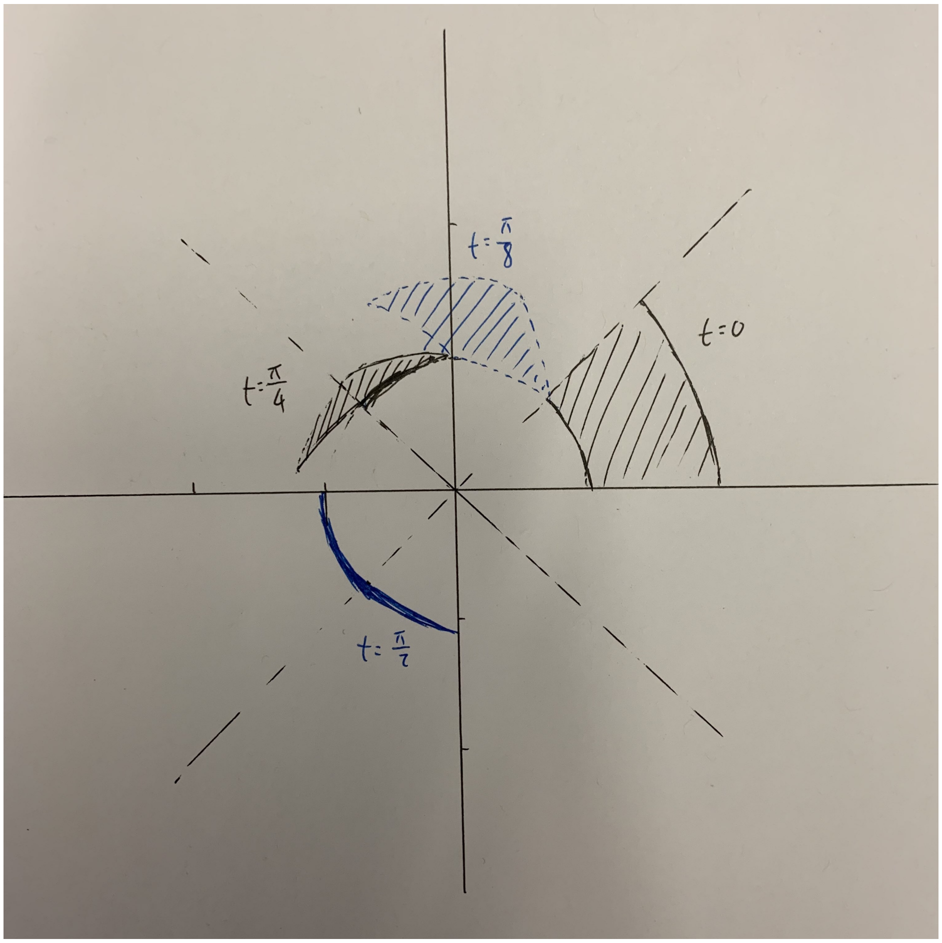
For $\langle r\rangle > \frac{1}{2}$, this quantity is negative, so phase space area will decrease. Since the box has $r > 1$, its area will decrease. We can compute again,

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} &(0.22)\\ &= \frac{x^2(1 - x^2 - y^2) + y^2(1 - x^2 - y^2)}{r} &(0.23)\\ &= \frac{r^2(1 - r^2)}{r} &(0.24)\\ &= r - r^3. &(0.25)\end{aligned}$$

For $r \geq 1$, this quantity is $\dot{r} \leq 0$, so the radius will decrease. However, it will never become smaller than $r = 1$ since $r = 1$ is a nullcline, i.e. $\dot{r} = 0$ when $r = 1$. Therefore, we can be guaranteed that the phase space area will always be decreasing. For completeness, we can also compute,

$$\begin{aligned} \dot{\theta} &= \frac{\dot{y}x - y\dot{x}}{r^2} &(0.26)\\ &= \frac{x^2(1 + x^2 + y^2) + xy(1 - x^2 - y^2) + y^2(1 + x^2 + y^2) - xy(1 - x^2 - y^2)}{r^2} &(0.27)\\ &= 1 + r^2, &(0.28)\end{aligned}$$

which is the same as before! We can plot out how the box changes over time below,




One important observation is that while it's true that the higher the radius, the faster it will rotate, note that because the radius will all approach $r = 1$, we eventually get a segment with near zero area and almost constant arclength rotating at a constant

$$\dot{\theta} = 2. \tag{0.29}$$

Overall, the phase volume decreases for this box. Here, the attractor is clearly the closed circle at $r = 1$. We will prove that this is true.

(a) $A$ is an invariant set because if we have initial conditions $r = 1, \theta = \theta_0$, then we have $\dot{r} = 0$, so we will always stay at $r = 1$.

(b) $A$ attracts an open set of initial conditions since if $r > 1$, we have shown that $\dot{r} < 0$, so the radius will decrease until it becomes arbitrarily close to $r = 1$ (but because of existence and uniqueness, will never reach $r = 1$). Similarly, if $0 < r < 1$, we have

$$\dot{r} = r - r^3 > 0, \tag{0.30}$$

so the radius will increase until it becomes arbitrarily close to $r = 1$. Therefore, consider the open set $(1 - \delta, 1 + \delta) \times [0, 2\pi)$ represented in polar coordinates, with $0 < \delta < 1$. By the above argument, any trajectory that starts in this set will eventually approach $r = 1$.

(c) $A$ is minimal since consider any proper subset of $A$, which we will call $\tilde{A}$. Since $\tilde{A}$ is a proper subset, there will be at least one point $(\cos\theta_0, \sin\theta_0)$ that is not contained in $\tilde{A}$ for $0 \leq \theta_0 < 2\pi$. However, we have shown that any trajectory that starts at $r = 1$ will have a constant angular velocity of $\dot{\theta} = 2$, so it must reach $\theta_0$ in a finite time. Therefore, $\tilde{A}$ cannot be minimal.

Therefore, the limit cycle $r = 1$ is an attractor for the system, so we have a limit cycle attractor, and hence it is not a strange attractor.

## Strogatz 7.2.9

To answer this roblem, we first need to prove a theorem (Strogatz 7.2.5),

**Theorem**: A system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a gradient system if and only if $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

*Proof.* First suppose that the system is a gradient system, i.e. we can write

$$\dot{x} = -\frac{\partial V}{\partial x}$$
$$\dot{y} = -\frac{\partial V}{\partial y}.$$

Then,

$$\frac{\partial \dot{x}}{\partial y} - \frac{\partial \dot{y}}{\partial x} = -\frac{\partial^2 V}{\partial y \partial x} + \frac{\partial^2 V}{\partial x \partial y},$$

which is zero by Clairaut's's theorem. Next, suppose that $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$. Note that $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ is a conservative vector field (and thus a gradient system) if it is path independent. This is true if the curl

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \tag{0.31}$$

is zero, which is true, per our assumption. □

a) We have

$$\frac{\partial \dot{x}}{\partial y} = 1 + x^2, \qquad \frac{\partial \dot{y}}{\partial x} = -1 + 2y. \tag{0.32}$$

These partial derivatives are not equal to each other, so the system is not conservative.

b) We have

$$\frac{\partial \dot{x}}{\partial y} = \frac{\partial \dot{y}}{\partial x} = 0, \tag{0.33}$$

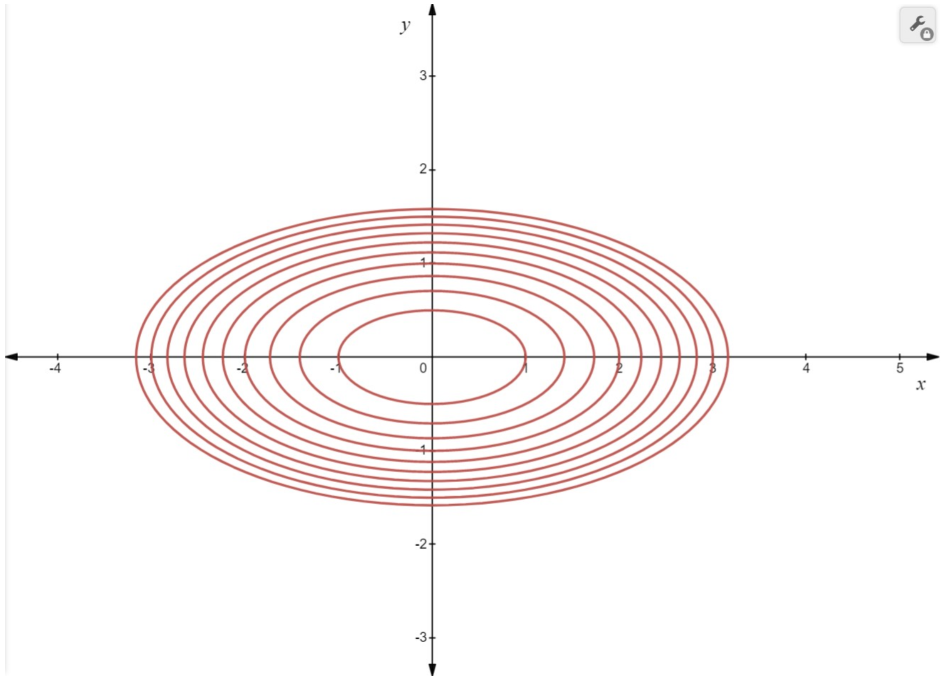
so the system is conservative. The potential function that leads to this can be found by integrating,

$$x = x^2 + f(y) \tag{0.34}$$
$$y = 4y^2 + f(x), \tag{0.35}$$

which leads to

$$V(x) = -x^2 - 4y^2. \tag{0.36}$$

The equipotential lines are shown in the below figure. The level curves are $V = -1, -2, -3, \ldots, -10$ with $V = -1$ corresponding to the smallest ellipse and $V = -10$ corresponding to the largest ellipse.



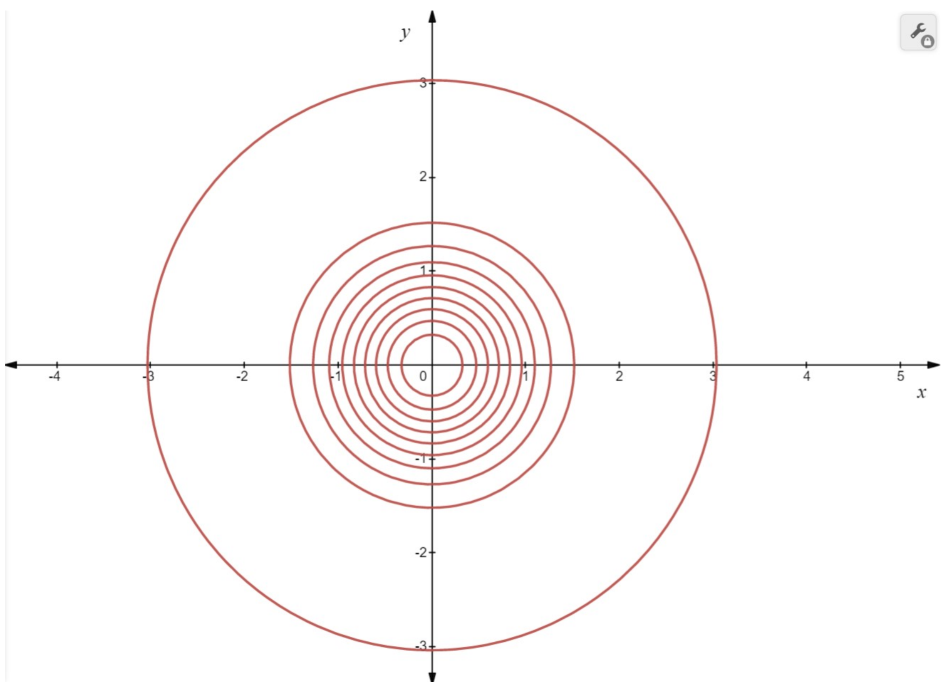

c) We have

$$\frac{\partial \dot{x}}{\partial y} = -2x(2y)e^{x^2+y^2}, \qquad \frac{\partial \dot{y}}{\partial x} = -2y(2x)e^{x^2+y^2}. \tag{0.37}$$

These partial derivatives are equal to each other, so the system is conservative. We can recover the potential by immediately recognizing that

$$V = e^{x^2+y^2}. \tag{0.38}$$

The equipotential lines are shown in the below figure. The level curves are $V = -0.0001, -0.1, -0.2, -0.3, \ldots, -0.9$. The level curve $V = -0.0001$ corresponds to the largest circle and the level curve $V = -0.9$ corresponds to the smallest circle.



## Strogatz 8.1.13

a) We have the system

$$\frac{dn}{dt} = GnN - kn \tag{0.39}$$
$$\frac{dN}{dt} = -GnN - fN + p \tag{0.40}$$

Consider

$$x = \frac{nG}{f}, \qquad y = \frac{NG}{f}, \qquad \tau = ft.$$

The first equation becomes

$$\frac{f}{G} \cdot f \cdot \frac{d(nG/f)}{d(ft)} = G \cdot \frac{xf}{G} \cdot \frac{yf}{G} - k \cdot \frac{xf}{G}$$
$$\implies \frac{dx}{d\tau} = xy - \frac{k}{f}x$$

and the second equation becomes

$$\frac{f}{G} \cdot f \cdot \frac{d(NG/f)}{d(ft)} = -G \cdot \frac{xf}{G} \cdot \frac{yf}{G} - f \cdot \frac{yf}{G} + p$$
$$\implies \frac{dx}{d\tau} = -xy - y + \frac{pG}{f^2}.$$

If we make the additional substitution $\alpha = \frac{k}{f}$ and $\beta = \frac{pG}{f^2}$, we get

$$\frac{dx}{d\tau} = xy - \alpha x \tag{0.41}$$
$$\frac{dy}{d\tau} = -xy - y + \beta. \tag{0.42}$$

Note that $\alpha > 0$ but $\beta$ can take on any sign since $p$ can take on any sign.

b) Fixed points are given when $\dot{x} = 0$, i.e.

$$xy - \alpha x = 0 \implies x = 0 \text{ OR } y = \alpha \tag{0.43}$$

and when $\dot{y} = 0$, i.e.

$$-xy - y + \beta = 0. \tag{0.44}$$

There are two cases here,

- Case 1: $x = 0$. Then this second equation becomes

$$-y + \beta = 0 \implies y = \beta. \tag{0.45}$$

- Case 2: $y = \alpha$. This second equation becomes

$$-x\alpha - \alpha + \beta = 0 \implies x = \frac{\beta}{\alpha} - 1. \tag{0.46}$$

The two fixed points are

$$(0, \beta), \left(\frac{\beta}{\alpha} - 1, \alpha\right). \tag{0.47}$$

Note that the second fixed point only exists in the physical world if $\beta \geq \alpha$, since we only positive values of $x, y$ correspond to meaningful physical states. Also note that at $\beta = \alpha$, these two fixed points coincide, suggesting a possible transcritical bifurcation here.

The Jacobian is given by

$$J = \begin{pmatrix} y-\alpha & x \\ -y & -x-1 \end{pmatrix}. \tag{0.48}$$

At the first fixed point, we have

$$J_0 = \begin{pmatrix} \beta-\alpha & 0 \\ -\beta & -1 \end{pmatrix}. \tag{0.49}$$

The determinant is $\Delta_0 = \alpha - \beta$ and the trace is $T_0 = \beta - \alpha - 1 = -\Delta_0 - 1$. We have several cases:

- Case 1: $\Delta_0 < 0$ : Linearization predicts a saddle point.
- Case 2: $\Delta_0 \geq 0$ : Linearization predicts a stable fixed point. The type of fixed point, according to linearization at least, will vary depending on the value of $\Delta_0$, but the actual type can only be determined via its nonlinear behavior. The fact that it will remain stable though is clear.

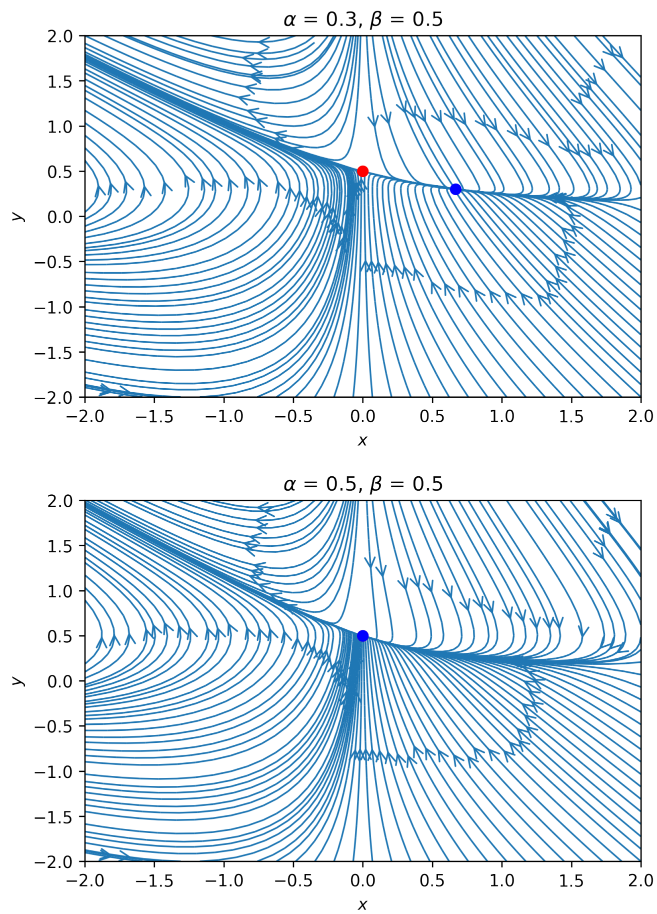
At the second fixed point, we have

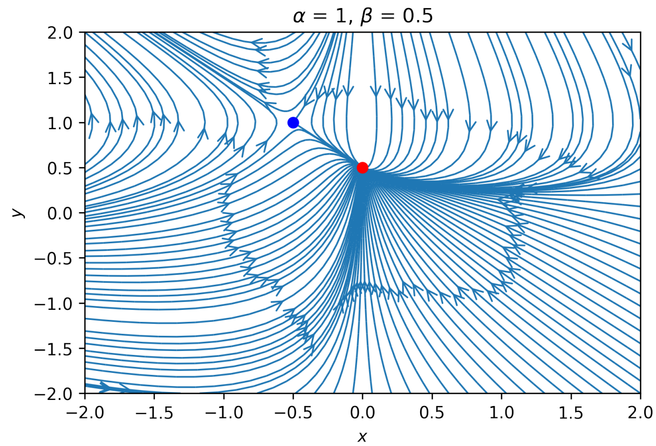
$$J_1 = \begin{pmatrix} 0 & \dfrac{\beta}{\alpha}-1 \\ -\alpha & -\dfrac{\beta}{\alpha} \end{pmatrix}. \tag{0.50}$$

The determinant is $\Delta_1 = \beta - \alpha$ and the trace is $T_1 = -\dfrac{\beta}{\alpha}$. We have several cases:

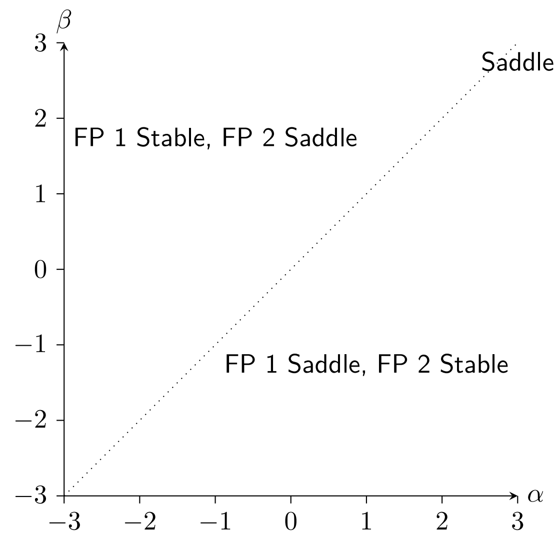
- Case 1: $\Delta_1 < 0$ : Linearization predicts a saddle point.
- Case 2: $\Delta_1 \geq 0$ : Linearization predicts a stable fixed point since $\beta > \alpha > 0$, so $T_1 < 0$. Similar to the other fixed point, the type of fixed point will be determined by its nonlinear behavior and at this point, we don't care about the specifics, only that it is stable.

c) See the attached plot. The fixed points are colour coded. Notice that the stability swaps.






d) See the below graph. Note that the dotted line represents the transcritical bifurcation where both fixed points merge into a single saddle point. Here, "FP" stands for fixed point.




Note that $\Delta_1 = -\Delta_0$. This means that when one fixed point is a saddle, the other fixed point is stable. At $\alpha = \beta$, they coincide and as these parameters continue to change, their stabilities flip. Therefore, we have a transcritical bifurcation.

## Strogatz 9.2.1

a) Suppose that $r > 1$ such that we have the fixed points $x^* = y^* = \pm\sqrt{b(r-1)}, z^* = r - 1$. Recall that the Lorentz equations are

$$\dot{x} = \sigma(y - x) \tag{0.51}$$
$$\dot{y} = rx - y - xz \tag{0.52}$$
$$\dot{z} = xy - bz. \tag{0.53}$$

It has a Jacobian

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}, \tag{0.54}$$

which at our positive fixed point, becomes

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}, \tag{0.55}$$

whose characteristic polynomial is given by

$$\text{char}(J) = \det(J - \lambda I) = \det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b-\lambda \end{pmatrix} \tag{0.56}$$

$$= -2br\sigma - br\lambda - b\sigma\lambda + 2b\sigma - b\lambda^2 - \sigma\lambda^2 - \lambda^3 - \lambda^2 \tag{0.57}$$

$$= -\lambda^3 - \lambda^2(1+\sigma+b) - \lambda b(\sigma + r) - 2b\sigma(r-1). \tag{0.58}$$

If we picked the negative fixed point instead, we have

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{pmatrix} \tag{0.59}$$

and

$$\text{char}(J) = \det(J - \lambda I) = \det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b-\lambda \end{pmatrix} \tag{0.60}$$

$$= -2br\sigma - br\lambda - b\sigma\lambda + 2b\sigma - b\lambda^2 - \sigma\lambda^2 - \lambda^3 - \lambda^2 \tag{0.61}$$

$$= -\lambda^3 - \lambda^2(1+\sigma+b) - \lambda b(\sigma + r) - 2b\sigma(r-1), \tag{0.62}$$

which is the same result as expected, since the Lorentz system is symmetric across the sign change $(x, y) \mapsto (-x, -y)$. This derived characteristic equation is exactly negative of the one we were tasked to find, which is fine, since what we really care about are the roots that this polynomial gives.

b) For this question, we will write our characteristic equation in the following form,

$$\lambda^3 + \lambda^2(1+\sigma+b) + \lambda b(\sigma + r) + 2b\sigma(r-1) = 0. \tag{0.63}$$

Let $\lambda = i\omega$, with $\omega$ real and plug it into the characteristic equation. We get,

$$-i\omega^3 + -\omega^2(1+\sigma+b) + i\omega b(\sigma + r) + 2b\sigma(r-1) = 0. \tag{0.64}$$

Let $r = r_H = \dfrac{\sigma^2 + b\sigma + 3\sigma}{\sigma - b - 1}$. Then,

$$\sigma + r = \frac{\sigma^2 - b\sigma - \sigma}{\sigma - b - 1} + \frac{\sigma^2 + b\sigma + 3\sigma}{\sigma - b - 1} \tag{0.65}$$

$$= 2\sigma \left( \frac{\sigma + 1}{\sigma - b - 1} \right) \tag{0.66}$$

and

$$r - 1 = \frac{\sigma^2 + b\sigma + 3\sigma}{\sigma - b - 1} - \frac{\sigma - b - 1}{\sigma - b - 1} \tag{0.67}$$

$$= \frac{\sigma^2 + (b+2)\sigma + b + 1}{\sigma - b - 1} \tag{0.68}$$

$$= \frac{(\sigma + b + 1)(\sigma + 1)}{\sigma - b - 1}. \tag{0.69}$$

We then have,

$$i\omega b(\sigma + r) + 2b\sigma(r-1) = i \cdot 2\sigma\omega b \left( \frac{\sigma + 1}{\sigma - b - 1} \right) + 2b\sigma \left( \frac{\sigma + 1}{\sigma - b - 1} \right) (\sigma + b + 1) \tag{0.70}$$

$$= 2b\sigma \left( \frac{\sigma + 1}{\sigma - b - 1} \right) (\omega i + \sigma + b + 1) \tag{0.71}$$

$$= 2b\sigma \left( \frac{\sigma + 1}{\sigma - b - 1} \right) (\omega i + K), \tag{0.72}$$

where we set $K = \sigma + b + 1$, Then the characteristic equation becomes

$$-i\omega^3 - K\omega^2 + 2b\sigma\left(\frac{\sigma+1}{\sigma-b-1}\right)(\omega i + K) = 0. \tag{0.73}$$

If this is equal to zero, both the real and imaginary components need to be zero. This gives,

$$0 = -\omega^3 + 2b\sigma\omega \cdot \frac{\sigma+1}{\sigma-b-1} \tag{0.74}$$

$$0 = -K\omega^2 + 2b\sigma K \cdot \frac{\sigma+1}{\sigma-b-1}, \tag{0.75}$$

which can be further reduced to

$$0 = -\omega^2 + 2b\sigma \cdot \frac{\sigma+1}{\sigma-b-1} \tag{0.76}$$

$$0 = K\left(-\omega^2 + 2b\sigma \cdot \frac{\sigma+1}{\sigma-b-1}\right), \tag{0.77}$$

so these two equations are actually the same! If the first equation is satisfied, i.e.

$$\omega^2 = 2b\sigma \cdot \frac{\sigma+1}{\sigma-b-1}, \tag{0.78}$$

then the second equation is automatically satisfied too. Note that if we want to satisfy the requirement that $\omega$ is real, we need $\omega^2 \geq 0$. We already know that $b, \sigma > 0$, so this is true if and only if $\sigma - b - 1 > 0 \iff \sigma \geq b + 1$. To get the strict requirement $\sigma > b + 1$, we note that $r$ is finite, so the denominator $\sigma - b - 1$ cannot be equal to zero. Since this rules out $\sigma \neq b - 1$, we are left with

$$\sigma > b + 1. \tag{0.79}$$

c) Recall that Vieta's formula tells us that for any cubic

$$P(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{0.80}$$

the sum of its roots are given by

$$r_1 + r_2 + r_3 = -\frac{a_2}{a_3}. \tag{0.81}$$

We already know two of the roots, $r_1 = +i\omega$ and $r_2 = -i\omega$. Therefore, the third root (i.e. eigenvalue) is

$$\lambda_3 = -\frac{1+\sigma+b}{1} = -\sigma - b - 1. \tag{0.82}$$

# Quadratic Map (Strogatz 10.3.4)

(a) Fixed points satisfy the property $x^* = (x^*)^2 + c = f(x^*)$ This gives the quadratic equation,

$$x^* = \frac{1 \pm \sqrt{1-4c}}{2}, \tag{0.83}$$

which only exists for $c \leq \frac{1}{4}$. To determine stability, we can compute the multiplier

$$|\lambda| = |f'(x^*)| = 2|x^*| = |1 \pm \sqrt{1-4c}|. \tag{0.84}$$

For $c < \frac{1}{4}$, the larger fixed point will have $|\lambda| > 1$, so it is always unstable. The smaller fixed point will have $|\lambda| < 1$, so it will be stable, so long as

$$|1 - \sqrt{1-4c}| < 1 \implies 1 - \sqrt{1-4c} > -1 \implies c > -\frac{3}{4}. \tag{0.85}$$

Therefore, if $c > -\frac{3}{4}$, then the smaller fixed point is stable. If $c < -\frac{3}{4}$, then the smaller fixed point is unstable. If $c = -\frac{3}{4}$, we need to look at higher order terms. We have,

$$x^* + \eta_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + \frac{1}{2}f''(x^*)\eta_n^2 \tag{0.86}$$

$$\implies \eta_{n+1} = \lambda\eta_n + \eta_n^2 \tag{0.87}$$

$$\implies \eta_{n+1} = -\eta_n + \eta_n^2. \tag{0.88}$$

If $\eta_n > 0$, then we have a stable fixed point. If $\eta_n < 0$, then we have an unstable fixed point. Therefore, $c = -\frac{3}{4}$ corresponds to a saddle point.

For $c = \frac{1}{4}$, we have a single fixed point at $x^* = \frac{1}{2}$ and $\lambda = 1$, we need to look at higher order terms. We have,

$$x^* + \eta_{n+1} = f(x^* + \eta) = f(x^*) + f'(x^*)\eta_n + \frac{1}{2}f''(x^*)\eta_n^2 \tag{0.89}$$

$$\implies \eta_{n+1} = \eta_n + \eta_n^2. \tag{0.90}$$

This is semi-stable since for positive $\eta_n$, this map will cause the deviation to grow larger but for negative $\eta_n$, this map will cause the deviation to grow smaller.

(b) From the previous discussion, we have a saddle-node bifurcation at $c = \frac{3}{4}$, where two fixed points join at $x^* = \frac{1}{2}$ and annihilate each other.

We also have a flip bifurcation at $c = -\frac{3}{4}$, where the stability of the smaller fixed point changes from stable to unstable.

(c) Recall that a 2-cycle exists iff there are two points $p, q$ such that $f(p) = q$ and $f(q) = p$. This condition gives the functional equation

$$f(f(p)) = p. \tag{0.91}$$

For the quadratic map, we have

$$\left(p^2 + c\right)^2 + c = p \tag{0.92}$$

$$\implies p^4 + 2p^2c - p + c^2 + c = 0. \tag{0.93}$$

Clearly, two solutions are $p = \pm x^*$, so this allows us to factor out $p^2 - p + c$, such that

$$(p^2 - p + c)(p^2 + p + c + 1) = 0, \tag{0.94}$$

which gives

$$p = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2} = \frac{-1 \pm \sqrt{-3 - 4c}}{2}. \tag{0.95}$$

This requires $c < \frac{3}{4}$. Note that we can't have $c = \frac{3}{4}$, because then we would $p = q$. But $p = q$ corresponds to a fixed point, not a 2-cycle. The multiplier is

$$\lambda = \frac{d}{dx}\bigg|_{x=p} f(f(x)) = f'(q)f'(p) = 4pq. \tag{0.96}$$

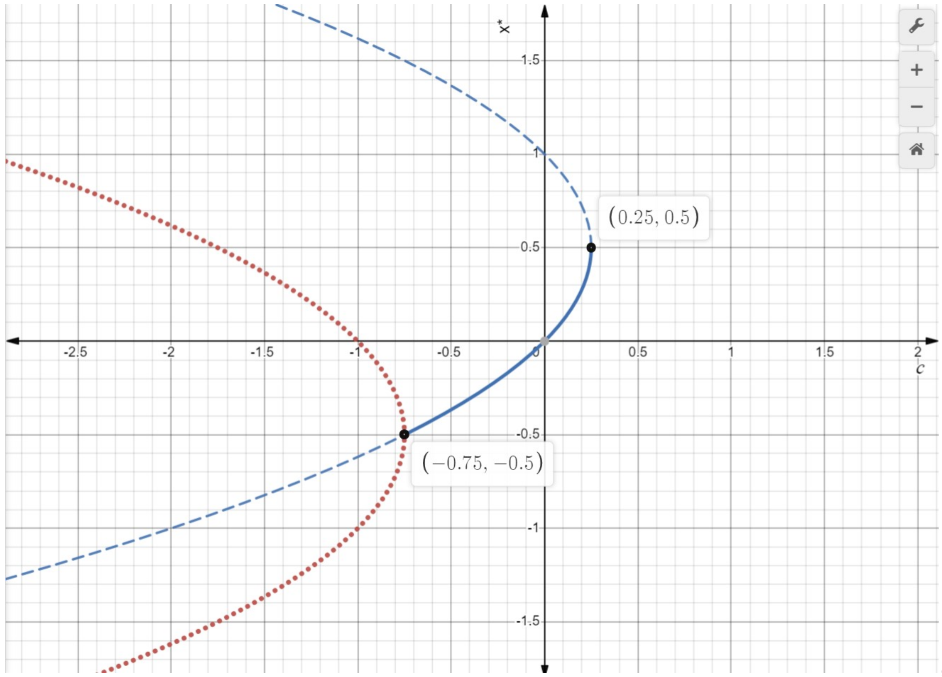
Recall that $p, q$ are the roots of

$$x^2 + x + c + 1, \tag{0.97}$$

and it is known the product of the roots of a quadratic is the constant term divided by the leading coefficient, so $pq = c+1$. Therefore,

$$|\lambda| < 1 \implies 4|c+1| < 1 \implies -\frac{5}{4} < c < -\frac{3}{4} \tag{0.98}$$

leads to a stable 2-cycle. A super-stable 2-cycle occurs when $\lambda = 0$, or when $4(c+1) = 0 \implies c = -1$. The bifurcation diagram is seen below.

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where the horizontal axis is $c$ and the vertical axis is $x^*$. Note that dashed lines represent unstable fixed points, solid lines represent stable fixed points, and dotted lines represent stable 2-cycles.